Exercise Set I, Algorithms II

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 *(easy)* Show that, given a matroid $\mathcal{M} = (E, \mathcal{I})$ and a weight function $w : E \to \mathbb{R}$, GREEDY (as defined in the lecture notes) always returns a base of the matroid.

Solution: Recall that a *base* of a matroid is an independent set of maximum cardinality. Let $B \in \mathcal{I}$ be a base of $\mathcal{M}$. Suppose towards a contradiction that the output $S \in \mathcal{I}$ of GREEDY is not a base of $\mathcal{M}$. Then $|S| < |B|$, and, by the second axiom of matroids, there exists some $e_b \in B \setminus S$ such that $(S \cup \{e_b\}) \in \mathcal{I}$. Let S' be the subset of elements in S that were considered before e_b by GREEDY. In other words, S' was the partial solution of GREEDY just before it considered e_b. By the first axiom of matroids, $S' \cup \{e_b\} \in \mathcal{I}$ because $S' \cup \{e_b\} \subseteq S \cup \{e_b\}$. Thus GREEDY should have added e_b to its solution S in Step 4, which is a contradiction.

Alternative solution. A base is a subset of maximal cardinality. Suppose that `Greedy` outputs $S \in \mathcal{I}$ but there exists $S^* \in \mathcal{I}$ such that $S \subsetneq S^*$ (thus S is not maximal). Let $e \in S^* \setminus S$. Now consider the current returning set F when `Greedy` observes e. Since $F + e \subseteq S^*$, we conclude from the first axiom that this set is independent. Thus we reach a contradiction because `Greedy` should have added e to F.

2 Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and a weight function $w : E \to \mathbb{R}$, GREEDY for matroids returns a base $S = \{s_1, s_2, \ldots, s_k\}$ of maximum weight. As noted in the lecture notes, any base consists of the same number, say k, of elements (which is said to be the rank of the matroid). We further assume that the elements of S are indexed so that $w(s_1) \geq w(s_2) \geq \cdots \geq w(s_k)$.

Let $S_\ell = \{s_1, \ldots, s_\ell\}$ be the subset of S consisting of the ℓ first elements, for $\ell = 1, \ldots, k$. Then prove that

$$w(S_\ell) = \max_{T \in \mathcal{I} : |T| = \ell} w(T) \text{ for all } \ell = 1, \ldots, k.$$

In other words, GREEDY does not only returns a base of maximum weight but the "prefixes" are maximum weight sets of respective cardinalities.

Solution: Consider some $\ell = 1, \ldots, k$ and define $\mathcal{I}_\ell = \{I \in \mathcal{I} : |I| \leq \ell\}$. There are two key observations:

- $\mathcal{M}_\ell = (E, \mathcal{I}_\ell)$ is a matroid called the truncated matroid of M.

- Greedy has an identical execution on $\mathcal{M}_\ell$ as for $\mathcal{M}$ until ℓ elements are selected.

From these properties (and the fact that Greedy works for matroids and thus for $\mathcal{M}_\ell$) we have that Greedy returns the max-weight base $S = \{s_1, \ldots, s_\ell\}$ of $\mathcal{M}_\ell$. In other words,

$$w(S_\ell) = \max_{T \in \mathcal{I}_\ell : |T| = \ell} w(T) = \max_{T \in \mathcal{I} : |T| = \ell} w(T)$$

as required.

3 *(easy)* Recall that a matroid $\mathcal{M} = (E, \mathcal{I})$ is a partition matroid if E is partitioned into *disjoint* sets $E_1, E_2, ..., E_\ell$ and

$$\mathcal{I} = \{X \subseteq E : |E_i \cap X| \leq k_i \text{ for } i = 1, 2, ..., \ell\}.$$

Verify that this is indeed a matroid.

Solution:
We show that $\mathcal{M}$ satisfies the two axioms I_1 and I_2 for matroids.

- Take any $A \in \mathcal{I}$ and let $B \subseteq A$. Then $|E_i \cap A| \leq k_i$ for all $i = 1, \ldots, \ell$. Clearly, all the inequalities $|E_i \cap B| \leq k_i$ also hold as $B \subseteq A$. Thus $B \in \mathcal{I}$, and Axiom I_1 is satisfied.
- Let $A, B \in \mathcal{I}$ and suppose that $|A| > |B|$. For all $i = 1, \ldots, \ell$, consider the sets $E_i \cap A$ and $E_i \cap B$. Since $|A|$ is strictly greater than $|B|$, there exists an index $j \in \{1, \ldots, \ell\}$ such that $|E_j \cap A| > |E_j \cap B|$. This implies that $|E_j \cap B| \leq |E_j \cap A| - 1 \leq k_j - 1$. Choose an element $e \in (E_j \cap A) \setminus (E_j \cap B) \subseteq A$ and add it to B. The new set $B \cup \{e\}$ satisfies $|E_j \cap (B \cup \{e\})| \leq k_j$. Clearly, the new set also satisfies all the remaining inequalities because $|E_i \cap (B \cup \{e\})| = |E_i \cap B|$ for $i \neq j$ (note that $e \notin E_i$ for $i \neq j$ because $e \in E_j$ and $E_1, E_2, \ldots, E_\ell$ are disjoint). Thus $B \cup \{e\} \in \mathcal{I}$, and Axiom I_2 is satisfied.

4 *(half a *)* Consider a bipartite graph $G = (V, E)$ where V is partitioned into A and B. Let $(A, \mathcal{I})$ be the matroid with ground set A and

$$\mathcal{I} = \{A' \subseteq A : \ G \text{ has a matching in which every vertex of } A' \text{ is matched}\}.$$

Recall that we say that a vertex is matched by a matching M if there is an edge in M incident to v. Show that $(A, \mathcal{I})$ is indeed a matroid by verifying the two axioms.

Solution: We need to verify the two axioms:

(I_1) Consider a set $A' \in \mathcal{I}$ and a subset $A'' \subseteq A'$. The matching that match every vertex in A' also matches every vertex in A'' so $A'' \in \mathcal{I}$ as required.

(I_2) Consider two sets $A_1, A_2 \in \mathcal{I}$ with $|A_1| < |A_2|$. Let M_1 and M_2 be the two matchings that matches all vertices in A_1 and A_2, respectively. Assume, w.l.o.g. that $|M_1| = |A_1|$ and $|M_2| = |A_2|$. Note that these matchings are guaranteed to exist since $A_1, A_2 \in \mathcal{I}$. Now the graph $(V, M_1 \cup M_2)$ has a matching of cardinality at least $|M_2|$. This means that there is an augmenting path P in $(V, M_1 \cup M_2)$ with respect to the matching M_1. If we let $M = M_1 \Delta P$ then M matches all vertices in A_1 plus one more vertex v from A since $|M| = |M_1| + 1$. This vertex has to be from A_2 since $A_1 \cup A_2$ are the only vertices of A that are incident to any edges in graph $(V, M_1 \cup M_2)$. It follows that $v \in A_2 \setminus A_1$ and $A_1 + v \in \mathcal{I}$ as required.

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5a *(*)* Consider a family $\mathcal{F}$ of subsets of the ground set E that satisfies: if $X, Y \in \mathcal{F}$ then either $X \cap Y = \emptyset$ (they are disjoint), $X \subseteq Y$ (X is a subset of Y), or $Y \subseteq X$ (Y is a subset of X). Show that for any positive integers $\{k_X\}_{X\in\mathcal{F}}$ (one for each set in $\mathcal{F}$) we have that $\mathcal{M} = (E, \mathcal{I})$ is a matroid, where

$$\mathcal{I} = \{S \subseteq E : |S \cap X| \leq k_X \text{ for every } X \in \mathcal{F}\}\,.$$

Such a matroid is called a laminar matroid.

Solution: (There are various proof of this fact. One is explained below.)

We show that $\mathcal{M}$ satisfies the two axioms I_1 and I_2 for matroids.

- Let $\emptyset \neq S \in \mathcal{I}$ and let $x \in S$. To show that $\mathcal{M}$ satisfies Axiom I_1, let $T \subsetneq S$. Since $S \in \mathcal{I}$, we have $|T \cap X| \leq |S \cap X| \leq k_X$ for each $X \in \mathcal{F}$. Hence, $T \in \mathcal{I}$.

- Let $A, B \in \mathcal{I}$ such that $|A| > |B|$. To show that $\mathcal{M}$ satisfies Axiom I_2, we show that there exist an element $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.

  Suppose that $|B \cap X| < k_X$ (with strict inequality) for all $X \in \mathcal{F}$. In this case, we choose any $x \in A \setminus B$ and add it to B, and the resulting set $B \cup \{x\}$ is in $\mathcal{I}$.

  Now suppose that $|B \cap X| = k_X$ for some set $X \in \mathcal{F}$. Then, we cannot add any $x \in A \cap X$ to B as it would violate the constraint for X ($|(B \cup \{x\}) \cap X| > k_X$ for such an x). However notice that $|A \cap X|$ can also have at most k_X elements. Thus A has at least $|A| - k_X$ elements outside X and B has exactly $|B| - k_X$ elements outside X. Since $|A| > |B|$ we have $|A| - k_X > |B| - k_X$, and consequently, we have more elements in $A \setminus X$ than in $B \setminus X$. We generalize this idea formally below.

  Let $\mathcal{F}^\star = \{X \in \mathcal{F} : |B \cap X| = k_X\}$ be the collection of sets in $\mathcal{F}$ for which the the constraints are satisfied with equality. By our assumption above, $\mathcal{F}^\star$ is non-empty. Let Y_1 be the largest set in $\mathcal{F}^\star$. Let Y_2 be the next largest set in $\mathcal{F}^\star$ that is disjoint with Y_1. After Y_i is selected, let Y_{i+1} be the next largest set in $\mathcal{F}^\star$ that is disjoint from each Y_j for $j = 1, 2, \ldots, i$. Stop this procedure when no more such sets can be selected, and let Y_m be the last selected set. Any of the remaining sets in $\mathcal{F}^\star$ is completely contained inside one of the sets $Y_1, Y_2, \ldots Y_m$ (why?). Let $Y = \cup_{i=1}^m Y_i$. We show that the number of elements in $A \setminus Y$ is more than the number of elements in $B \setminus Y$. We have

  $$|A| = |(A \cap Y) \cup (A \setminus Y)| = \left|\left(\cup_{i\in[m]}(A \cap Y_i)\right) \cup (A \setminus Y)\right| = |A \setminus Y| + \sum_{i\in[m]} |A \cap Y_i|,$$

  and similarly, we have

  $$|B| = |B \setminus Y| + \sum_{i\in[m]} |B \cap Y_i|.$$

  But $|B \cap Y_i| = k_{Y_i}$ and $|A \cap Y_i| \leq k_{Y_i}$ for all $i = 1, \ldots, m$. Thus using $|A| > |B|$ we conclude that $|A \setminus Y| > |B \setminus Y|$. Choose an element $x \in (A \setminus Y) \setminus B$. By our selection of Y_i's, x is not in any of the sets in $\mathcal{F}^\star$, and therefore, adding it to B would not violate those constrains for sets in $\mathcal{F}^\star$. For all sets in $\mathcal{F} \setminus \mathcal{F}^\star$, the respective constraints have some slack and adding x to B would not violate those constraints either.

5b Argh! Buying the DVD rental shop was not such a great idea. After the explosion of more convenient streaming services, you are now forced to close your business venture. But what should you do with all your DVDs? To be exact, you have n DVDs and each one is placed in one of the following genres: action, comedy, drama, horror or adventure. As you are a very nice person, you decide to distribute these DVDs among your most loyal customers. You have m loyal customers and for each DVD i and customer j there is a positive weight $w(i,j)$ that models how interesting DVD i is for customer j. Your goal is to find an assignment of DVDs to loyal customers satisfying the following:

- Each DVD is assigned to at most one customer.
- Each customer receives at most 5 DVDs in total and no more than 2 DVDs of the same genre.
- The total weight (called the social welfare) of your assignment is maximized.

Show that the problem of distributing the DVDs as above can be formulated as that of finding a maximum weight independent set in the intersection of two matroids.

Solution: In this problem, we need to satisfy two conditions.

1. Each DVD is assigned at most one customer
2. Each customer receives at most 5 DVDs in total and no more than 2 DVDs of the same genre.

Let the customers be numbered $1, \ldots, m$ and DVDs be numbered $1, \ldots, n$. Let $E = \{(i,j) : i \in [m], j \in [n]\}$ be the set of possible edges, which will be the ground set for our matroids.

Let $S \subseteq E$ be any assignment of DVDs to customers that satisfies the two constraints.

Let $D_j = \{(i,j) : i \in [m]\}$ for all $j \in [n]$. In order to satisfy Condition 1, it is clear that, for all $j \in [n]$, $|S \cap D_j| \leq 1$. Hence, we define our first matroid as the following partition matroid.

$$\mathcal{I}_1 = \{S \subseteq E : |S \cap D_j| \leq 1 \text{ for all } j \in [n]\}\,.$$

For the second constraint, we use the result from **1a**. Let $G_1, \ldots G_5$ be a partition of $[n]$ corresponding to genres 'action', 'comedy', 'drama', 'horror' and 'adventure' respectively. For $i \in [n]$, let $C_i = \{(i,j) \in E : j \in [n]\}$ be the set of edges going from customer i to the set of DVDs. For $i \in [m], \ell \in [5]$, let $T_{i\ell} = \{(i,j) \in E : j \in G_\ell\}$ be the set of edges going from customer i to DVDs of genre G_ℓ. Note that C_i's are a partitioning of E and, for each i, $T_{i\ell}$'s are a partitioning of C_i.

If S satisfies Condition 2, it must be the case that, $|S \cap T_{i\ell}| \leq 2$ for all $\ell = 1, \ldots, 5$ and $|S \cap C_i| \leq 5$ for all $i \in [n]$.

Let $k_{T_{i\ell}} = 2$ for all $i \in [n], \ell \in [5]$ and let $k_{C_i} = 5$. Let $\mathcal{F} = \{T_{i\ell} : i \in [n], \ell \in [5]\} \cup \{C_i : i \in [m]\}$. Since C_i's are a partitioning of E and since for each i, $T_{i\ell}$'s are a partitioning of C_i, any $X, Y \in \mathcal{F}$ satisfies either $X \cap Y = \emptyset$ or $X \subseteq Y$ or $Y \subseteq X$. Thus, from **1a** the following is a matroid.

$$\mathcal{I}_2 = \{S \subseteq E : |S \cap X| \leq k_X \text{ for all } X \in \mathcal{F}\}\,.$$

From the above discussion it is clear that a solution S is feasible if and only if it is independent in both $\mathcal{I}_1$ and $\mathcal{I}_2$. Hence, the problem is equivalent to finding the *maximum weight independent set* in the intersection of the two matroids, $\mathcal{I}_1$ and $\mathcal{I}_2$.

6 **Spanning trees with colors.** Consider the following problem where we are given an edge-colored graph and we wish to find a spanning tree that contains a specified number of edges of each color:

Input: A connected undirected graph $G = (V, E)$ where the edges E are partitioned into k color classes $E_1, E_2, \ldots, E_k$. In addition each color class i has a target number $t_i \in \mathbb{N}$.

Output: If possible, a spanning tree $T \subseteq E$ of the graph satisfying the color requirements:

$$|T \cap E_i| = t_i \qquad \text{for } i = 1, \ldots, k.$$

Otherwise, i.e., if no such spanning tree T exists, output that no solution exists.

Design a polynomial time algorithm for the above problem. You should analyze the correctness of your algorithm, i.e., why it finds a solution if possible. To do so, you are allowed to use algorithms and results seen in class without reexplaining them.

Solution: We solve this problem using matroid intersection. First observe that if the summation of the t_i for $1 \leq i \leq k$ is not equal to $n-1$ then there is no feasible solution since we know that the number of edge in any spanning tree is exactly $n-1$. Therefore, we assume $\sum_{1 \leq i \leq k} t_i = n-1$. The ground set for both matroids that we use is the set of the edges E. First matroid that we use is the graphic matroid. The second matroid that we use is a partition matroid with following independent sets:

$$\mathcal{I} = \{F \subseteq E \mid |F \cap E_i| \leq t_i, \quad \text{for } 1 \leq i \leq k\}$$

As shown in class the both above defined matroids are indeed matroid. Now assume that F is the maximum size independent set the intersection of these two matroids (we saw in the class how we can find F). If $|F| < n-1$ it is not possible to find a solution for our problem, since any solution to our problem corresponds to a solution in the intersection of these two matroids of size $n-1$. Moreover, if $|F| = n-1$, than F is a spanning tree and $|F \cap E_i| \leq t_i$. Also, we know that $|F| = n-1$ and $\sum_{1 \leq i \leq k} t_i = n-1$ and E_i's are disjoint. Therefore $|F \cap E_i| = t_i$, so we get the desired solution.