Exercise Set XII

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually. **The problems are not ordered with respect to difficulty.**

1 Consider an undirected graph $G = (V, E)$. Show that the following linear program, that has a variable x_v for each $v \in V$, can be solved in polynomial time

$$\begin{aligned} \textbf{maximize} \quad & \sum_{v \in V} x_v \\ \textbf{subject to} \quad & \sum_{v \in S} x_v \leq |\delta(S)| \qquad \text{for every } S \subsetneq V \\ & x_v \geq 0 \end{aligned}$$

Recall that $\delta(S)$ is the set of edges crossing the cut S.

By Ellipsoid method this reduces to the following separation problem: Given x, design a polytime algorithm that verifies whether x is feasible or, if not, outputs a violated constraint.

Solution:

- To verify that x is non-negative is trivial to verify in polynomial time and if not simply output a violated inequality $x_v \geq 0$.
- It remains to deal with the exponential set of inequalities

$$\sum_{v \in S} x_v \leq |\delta(S)| \qquad \text{for every } S \subsetneq V.$$

- We can rewrite the above

$$0 \leq f(S) \qquad \text{for every } S \subsetneq V,$$

  where $f(S) = |\delta(S)| - \sum_{v \in S} x_v$.
- The key insight is that f is a submodular set function:
  - $|\delta(S)|$ is the cut function (as seen in class)

  - $-\sum_{v\in S} x_v$ is a modular function and thus trivially submodular (the values are independent)
  - The sum of two submodular functions are submodular as seen in class

- We can thus minimize f in polynomial time (since we can evaluate $f(S)$ in polynomial time). If the minimizer is ≥ 0 then x is feasible. Otherwise the minimizer S corresponds to a violated constraint if $S \neq V$.

- To deal with that we don't allow $S = V$ we can do n submodular function minimizations over the submodular functions $g_1, g_2, \ldots, g_n$ where the i-th function g_i is defined over the ground set $V \setminus \{v_i\}$ and $g(S) = f(S)$ for every $S \subseteq V \setminus \{v_i\}$. Here, we name the vertices so that $V = \{v_1, \ldots, v_n\}$.

2 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$. In class, we proved that the maximum eigenvalue equals 1.

Show that the maximum *absolute* value of an eigenvalue is at most 1. That is, for any eigenvalue λ of M, we have $|\lambda| \leq 1$.

Solution: Deferred to exercise set XIII.

3 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$ that is connected. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of M. Show that $\lambda_n = -1$ if and only if G is bipartite.

(Hint: to show $\lambda_n = -1$, we only need to find a vector x such that $Mx = -x$.)

Solution: Deferred to exercise set XIII.