Exercise Set XII, Advanced Algorithms 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually. **The problems are not ordered with respect to difficulty.**

1 Consider an undirected graph $G = (V, E)$. Show that the following linear program, that has a variable x_v for each $v \in V$, can be solved in polynomial time

$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{v \in V} x_v & \\ \textbf{subject to} & \displaystyle\sum_{v \in S} x_v \leq |\delta(S)| & \text{for every } S \subsetneq V \\ & \quad x_v \geq 0 & \end{array}$$

Recall that $\delta(S)$ is the set of edges crossing the cut S.

By Ellipsoid method this reduces to the following separation problem: Given x, design a polytime algorithm that verifies whether x is feasible or, if not, outputs a violated constraint.

2 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$. In class, we proved that the maximum eigenvalue equals 1.

Show that the maximum *absolute* value of an eigenvalue is at most 1. That is, for any eigenvalue λ of M, we have $|\lambda| \leq 1$.

3 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$ that is connected. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of M. Show that $\lambda_n = -1$ if and only if G is bipartite.

(Hint: to show $\lambda_n = -1$, we only need to find a vector x such that $Mx = -x$.)