Exercise Set XIII

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually. **The problems are not ordered with respect to difficulty.**

1 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$. In class, we proved that the maximum eigenvalue equals 1.

Show that the maximum *absolute* value of an eigenvalue is at most 1. That is, for any eigenvalue λ of M, we have $|\lambda| \leq 1$.

2 Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$ that is connected. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of M. Show that $\lambda_n = -1$ if and only if G is bipartite.

(Hint: to show $\lambda_n = -1$, we only need to find a vector x such that $Mx = -x$.)

3 In spectral graph theory, a popular matrix is the (normalized) Laplacian matrix. Its definition (for d-regular graphs) is as follows. Let M be the normalized adjacency matrix of a d-regular undirected graph $G = (V, E)$. The normalized Laplacian matrix is $L = I - M$.

3a Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of M. Show that $1 - \lambda_1 \leq 1 - \lambda_2 \leq \ldots \leq 1 - \lambda_n$ are the eigenvalues of L.

3b One reason for the popularity of the normalized Laplacian matrix is because its quadratic form $x^\top L x$ highlights the connection between cuts and eigenvectors. Indeed, verify the following identity

$$x^\top L x = \frac{1}{d} \sum_{\{i,j\} \in E} (x(i) - x(j))^2 .$$

Notice that if $x \in \{0, 1\}^n$, then the above identity says that $x^\top L x$ equals the number of edges cut by the set $S = \{i : x(i) = 1\}$ normalized by $\frac{1}{d}$.

4 Let $G = (V, E)$ be a d-regular undirected graph $G = (V, E)$. In this problem we shall analyze the *lazy* random walk on G:

- With probability 1/2: we stay at the current vertex
- With remaining probability 1/2: we go to a random neighbor (out of d possibilities).

4a Show that the smallest eigenvalue of the matrix corresponding to a lazy random walk is at least 0.

Some questions related to any subject of the course

5 Given a graph $G = (V, E)$ and an integer k, design a randomized algorithm that returns a coloring $c : V \to \{1, \ldots, k\}$ such that in expectation at least $\left(1 - \frac{1}{k}\right)$-fraction of the edges are *correctly colored.* An edge $e = \{u, v\}$ is correctly colored if it is not monochromatic, i.e., $c(u) \neq c(v)$.

6 Consider the matching problem in general graphs. Suppose that you are given a black-box polynomial-time algorithm $\mathcal{A}$ which, given a graph, returns TRUE if the graph has a perfect matching and NO otherwise[1] Explain how to use $\mathcal{A}$ in order to, in polynomial time,

a find a perfect matching in a graph,

b find a maximum cardinality matching in a graph.

[1] For example, the algorithm obtained by calculating the determinant of the Tutte matrix that we saw in class.