Exercise Set II

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 *(half a *)* Devise an algorithm for the following graph orientation problem:

Input: An undirected graph $G = (V, E)$ and capacities $k : V \to \mathbb{Z}$ for each vertex.

Output: If possible, an orientation of G such that each vertex $v \in V$ has in-degree at most $k(v)$.

An orientation of an undirected graph G replaces each undirected edge $\{u, v\}$ by either an arc (u, v) from u to v or by an (v, u) from v to u.

(Hint: reduce the problem to matroid intersection. You can also use bipartite matching...)

2 The first problem is difficult so you may want to skip that and solve (6b) assuming (6a) and then try (6a) if you have time.

2a *(*)* Consider a family $\mathcal{F}$ of subsets of the ground set E that satisfies: if $X, Y \in \mathcal{F}$ then either $X \cap Y = \emptyset$ (they are disjoint), $X \subseteq Y$ (X is a subset of Y), or $Y \subseteq X$ (Y is a subset of X). Show that for any positive integers $\{k_X\}_{X \in \mathcal{F}}$ (one for each set in $\mathcal{F}$) we have that $\mathcal{M} = (E, \mathcal{I})$ is a matroid, where

$$\mathcal{I} = \{S \subseteq E : |S \cap X| \leq k_X \text{ for every } X \in \mathcal{F}\}.$$

Such a matroid is called a laminar matroid.

2b Argh! Buying the DVD rental shop was not such a great idea. After the explosion of more convenient streaming services, you are now forced to close your business venture. But what should you do with all your DVDs? To be exact, you have n DVDs and each one is placed in one of the following genres: action, comedy, drama, horror or adventure. As you are a very nice person, you decide to distribute these DVDs among your most loyal customers. You have m loyal customers and for each DVD i and customer j there is a positive weight $w(i, j)$ that models how interesting DVD i is for customer j. Your goal is to find an assignment of DVDs to loyal customers satisfying the following:

- Each DVD is assigned to at most one customer.
- Each customer receives at most 5 DVDs in total and no more than 2 DVDs of the same genre.
- The total weight (called the social welfare) of your assignment is maximized.

Show that the problem of distributing the DVDs as above can be formulated as that of finding a maximum weight independent set in the intersection of two matroids.

3 **Spanning trees with colors.** Consider the following problem where we are given an edge-colored graph and we wish to find a spanning tree that contains a specified number of edges of each color:

Input: A connected undirected graph $G = (V, E)$ where the edges E are partitioned into k color classes $E_1, E_2, \ldots, E_k$. In addition each color class i has a target number $t_i \in \mathbb{N}$.

Output: If possible, a spanning tree $T \subseteq E$ of the graph satisfying the color requirements:

$$|T \cap E_i| = t_i \qquad \text{for } i = 1, \ldots, k.$$

Otherwise, i.e., if no such spanning tree T exists, output that no solution exists.

Design a polynomial time algorithm for the above problem. You should analyze the correctness of your algorithm, i.e., why it finds a solution if possible. To do so, you are allowed to use algorithms and results seen in class without reexplaining them.

4 For a bipartite graph, devise an efficient algorithm for finding an augmenting path P (if one exists). What is the total running time of the AugmentingPathAlgorithm explained in the second lecture?

5 You have just started your prestigious and important job as the Swiss Cheese Minister. As it turns out, different fondues and raclettes have different nutritional values and different prices:

Food	Fondue moitie moitie	Fondue a la tomate	Raclette	Requirement per week
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin B [mg/kg]	60	300	0.5	15 mg
Vitamin C [mg/kg]	30	20	70	4 mg
[price [CHF/kg]	50	75	60	—

Formulate the problem of finding the cheapest combination of the different fondues (moitie moitie & a la tomate) and Raclette so as to satisfy the weekly nutritional requirement as a linear program.

6 Consider the following linear program for finding a maximum-weight matching:

$$\begin{aligned} \text{Maximize} \quad & \sum_{e \in E} x_e w_e \\ \text{Subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

(This is similar to the perfect matching problem seen in the lecture, except that we have inequality constraints instead of equality constraints.) Prove that, for bipartite graphs, any extreme point is integral.

7 *(half a *)* Use the integrality of the bipartite perfect matching polytope (as proved in class) to show the following classical result:

> The edge set of a k-regular bipartite graph $G = (A \cup B, E)$ can in polynomial time be partitioned into k disjoint perfect matchings.

A graph is k-regular if the degree of each vertex equals k. Two matchings are disjoint if they do not share any edges.