# Exercise Set III

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

**1** You have just started your prestigious and important job as the Swiss Cheese Minister. As it turns out, different fondues and raclettes have different nutritional values and different prices:

| Food | Fondue moitie moitie | Fondue a la tomate | Raclette | Requirement per week |
|---|---|---|---|---|
| Vitamin A [mg/kg] | 35 | 0.5 | 0.5 | 0.5 mg |
| Vitamin B [mg/kg] | 60 | 300 | 0.5 | 15 mg |
| Vitamin C [mg/kg] | 30 | 20 | 70 | 4 mg |
| [price [CHF/kg] | 50 | 75 | 60 | — |

Formulate the problem of finding the cheapest combination of the different fondues (moitie moitie & a la tomate) and Raclette so as to satisfy the weekly nutritional requirement as a linear program.

**Solution:** We have a variable $x_1$ for moitie moitie, a variable $x_2$ for a la tomate, and a variable $x_3$ for Raclette. The linear program becomes

$$\begin{aligned}
\text{Minimize} \quad & 50x_1 + 75x_2 + 60x_3 \\
\text{Subject to} \quad & 35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5 \\
& 60x_1 + 300x_2 + 0.5x_3 \geq 15 \\
& 30x_1 + 20x_2 + 70x_3 \geq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

**2** Consider the following linear program for finding a maximum-weight matching:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{e \in E} x_e w_e \\
\text{Subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\
& x_e \geq 0 \quad \forall e \in E
\end{aligned}$$

(This is similar to the perfect matching problem seen in the lecture, except that we have inequality constraints instead of equality constraints.) Prove that, for bipartite graphs, any extreme point is integral.

**Solution:** We prove that all the extreme points are integral by contradiction. To that end, assume that there exists an extreme point $x^*$ that is not integral. Let $G = (V_1, V_2, E)$ be the given bipartite graph and let $E_f = \{e \in E \mid 0 < x^*_e < 1\}$. If $E_f$ contains a cycle, then the proof follows in the same way as the proof in the lecture notes. Therefore, we assume that $E_f$ does not contain any cycles. Consider any maximal path in $E_f$; let it have vertices $v_1, ..., v_k$ and edges $e_1, ..., e_{k-1}$. Choose any $\epsilon$ such that $0 < \epsilon < \min(x^*_{e_i}, 1 - x^*_{e_i} : i = 1, ..., k-1)$. Note that, since $E_f$ only contains edges that are fractional, such an $\epsilon$ exists. Let $y, z$ be the following two solutions to the linear program:

$$y = \begin{cases} x^*_e + \epsilon & \text{if } e \in \{e_1, e_3, e_5, e_7, ...\} \\ x^*_e - \epsilon & \text{if } e \in \{e_2, e_4, e_6, e_8, ...\} \\ x^*_e & \text{otherwise} \end{cases}$$

$$z = \begin{cases} x^*_e - \epsilon & \text{if } e \in \{e_1, e_3, e_5, e_7, ...\} \\ x^*_e + \epsilon & \text{if } e \in \{e_2, e_4, e_6, e_8, ...\} \\ x^*_e & \text{otherwise} \end{cases}$$

One can see that $x^* = \frac{y+z}{2}$.

We continue by showing that $y$ is a feasible solution to the linear program. One can see that for any vertex $v \in G$ except $v_1$ and $v_k$ we have $\sum_{e \in \delta(v)} y_e = \sum_{e \in \delta(v)} x^*_e$. So, we only need to show that the linear program constraint holds for $v_1$ and $v_k$. Let us first state two observations. First, by the definition of $\epsilon$, we have that $0 \leq x^*_{e_1} + \epsilon \leq 1$, $0 \leq x^*_{e_1} - \epsilon \leq 1$, $0 \leq x^*_{e_{k-1}} + \epsilon \leq 1$, and $0 \leq x^*_{e_{k-1}} - \epsilon \leq 1$. Second, since the path is maximal and $E_f$ does not contain any cycles, the degrees of $v_1$ and $v_k$ in $E_f$ are both one. Therefore $\sum_{e \in \delta(v_1)} y_e = y_{e_1}$ and $\sum_{e \in \delta(v_k)} y_e = y_{e_{k-1}}$. Putting together the previous two observations, we get that the linear program constraint also holds for $v_1$ and $v_k$, so $y$ is a feasible solution.

We can similarly show that $z$ is also a feasible solution. This shows that we can write $x^*$ as a convex combination of $y$ and $z$, which contradicts the fact that $x^*$ is an extreme point.

**3** *(half a *)* Use the integrality of the bipartite perfect matching polytope (as proved in class) to show the following classical result:

> The edge set of a $k$-regular bipartite graph $G = (A \cup B, E)$ can in polynomial time be partitioned into $k$ disjoint perfect matchings.

A graph is $k$-regular if the degree of each vertex equals $k$. Two matchings are disjoint if they do not share any edges.

**Solution:** We show how to find such $k$ disjoint perfect matchings in a $k$-regular bipartite graph in polynomial time.

Let $G_0 = (A \cup B, E)$ be a $k$-regular bipartite graph. Consider the LP for bipartite perfect matching on $G_0$. The LP is feasible because setting $x_e = 1/k$ for all $e \in E$ satisfies all the constraints (recall that each vertex of a $k$-regular graph is incident to exactly $k$ edges). Now we find an extreme point solution to the LP in polynomial time, and due to the integrality of such solutions, we get a valid perfect matching $M_1$.

Notice that $M_1$, being a perfect matching, forms a 1-regular sub-graph of G. Therefore, if we remove the matching $M_1$ from the original graph $G_0$, we get a new $(k-1)$-regular graph $G_1 = (A \cup B, E \setminus M)$.

Now we repeat the process $k$ times. Formally, at each iteration $i = 1, \dots, k$, we start with $(k - i + 1)$-regular graph $G_{i-1}$. By solving the bipartite perfect matching LP for $G_{i-1}$ to get an extreme point solution, we obtain a perfect matching $M_i$. We remove $M_i$ from $G_{i-1}$ to obtain a $(k - i)$-regular graph $G_i$, which is a sub-graph of $G_{i-1}$.

Since we remove the already found perfect matchings at each iteration, the $k$-perfect matchings $M_1, \dots, M_k$ are disjoint. Furthermore, since all graphs $G_1, \dots, G_{k-1}$ are sub-graphs of the original graph $G_0$, the matchings $M_1, \dots, M_k$ are all valid perfect matchings of $G_0$.

**4** *(*)* Consider the linear programming relaxation for minimum-weight vertex cover:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{v \in V} x_v w(v) \\ \text{Subject to} & x_u + x_v \geq 1 \quad \forall \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \qquad \forall v \in V \end{array}$$

In class, we saw that any extreme point is integral when considering bipartite graphs. For general graphs, this is not true, as can be seen by considering the graph consisting of a single triangle. However, we have the following statement for general graphs:

Any extreme point $x^*$ satisfies $x^*_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Prove the above statement.

**Solution:** Consider an extreme point $x^*$, and suppose for the sake of contradiction that $x^*$ is not half-integral, i.e., that there is an edge $e$ such that $x^*_e \notin \{0, \frac{1}{2}, 1\}$. We will show that $x^*$ is a convex combination of feasible points, contradicting that $x^*$ is an extreme point. Let $V^+ = \{v : \frac{1}{2} < x^*_v < 1\}$ and $V^- = \{v : 0 < x^*_v < \frac{1}{2}\}$. Note that $V^+ \cup V^- \neq \emptyset$, since $x^*$ is assumed to not be half-integral. Take $\epsilon > 0$ to be tiny, and define:

$$y^+_v = \begin{cases} x^*_v + \epsilon & \text{if } v \in V^+ \\ x^*_v - \epsilon & \text{if } v \in V^- \\ x^*_v & \text{otherwise} \end{cases}$$

$$y^-_v = \begin{cases} x^*_v - \epsilon & \text{if } v \in V^+ \\ x^*_v + \epsilon & \text{if } v \in V^- \\ x^*_v & \text{otherwise} \end{cases}$$

Note that $x^* = \frac{1}{2}y^+ + \frac{1}{2}y^-$.

It remains to verify that $y^+$ and $y^-$ are feasible solutions.

1. By selecting $\epsilon$ small enough, the boundary constraints $(0 \leq y^+_v \leq 1, 0 \leq y^-_v \leq 1)$ are satisfied.
2. Consider the constraints for the edges $e = \{u, v\} \in E$. If $x^*_u + x^*_v > 1$, the constraint remains satisfied by picking $\epsilon > 0$ small enough. If $x^*_u + x^*_v = 1$, then consider the following cases:
   - $u, v \notin V^+ \cup V^-$. In this case, $y^+_u + y^+_v = x^*_u + x^*_v = 1$.
   - $u \in V^+$; then $v \in V^-$. In this case, $y^+_u + y^+_v = x^*_u + \epsilon + x^*_v - \epsilon = 1$.
   - $u \in V^-$; then $v \in V^+$. In this case, $y^+_u + y^+_v = x^*_u - \epsilon + x^*_v + \epsilon = 1$.

So $y^+$ is a feasible solution. The same argument holds for $y^-$.

**5** *(*)* Consider an undirected graph $G = (V, E)$ and let $s \neq t \in V$. Recall that in the min $s, t$-cut problem, we wish to find a set $S \subseteq V$ such that $s \in S$, $t \notin S$ and the number of edges crossing the cut is minimized. Show that the optimal value of the following linear program equals the number of edges crossed by a min $s, t$-cut:

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{e \in E} y_e & \\
\textbf{subject to} & y_{\{u,v\}} \geq x_u - x_v & \text{for every } \{u,v\} \in E \\
& y_{\{u,v\}} \geq x_v - x_u & \text{for every } \{u,v\} \in E \\
& x_s = 0 & \\
& x_t = 1 & \\
& x_v \in [0,1] & \text{for every } v \in V
\end{array}$$

The above linear program has a variable $x_v$ for every vertex $v \in V$ and a variable $y_e$ for every edge $e \in E$.

*Hint: Show that the expected value of the following randomized rounding equals the value of the linear program. Select $\theta$ uniformly at random from $[0, 1]$ and output the cut $S = \{v \in V : x_v \leq \theta\}$.*

**Solution:** Let $OPT$ be the number of edges that cross a minimum $s, t$-cut, and let $OPT_{LP}$ be the value of the given LP. To show that $OPT = OPT_{LP}$, we show that $OPT_{LP} \leq OPT$ and $OPT_{LP} \geq OPT$.

Firstly let's prove that $OPT_{LP} \leq OPT$. Suppose that $S$ is an optimal cut $s, t$-cut. We have $s \in S$ and $t \notin S$. We will create a solution for the LP problem whose value equals cut size defined by $S$ and $E \setminus S$. Set $x_u = 0$ for all $u \in S$, and $x_v = 1$ for all $v \notin S$. Furthermore define

$$y_e = \begin{cases} 1 & \text{if } e \in \delta(S) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_e y_e = |\delta(S)| = OPT$. It remains to prove that the assignment to the variables $\{x_v\}_{v \in V}$, $\{y_e\}_{e \in E}$ is feasible:

- Consider any edge $\{u, v\}$. We need to verify that $y_{\{u,v\}} \geq x_u - x_v$ and $y_{\{u,v\}} \geq x_v - x_u$. In other words, that $y_{\{u,v\}} \geq |x_u - x_v|$.
  - If $\{u, v\} \in \delta(S)$ then one of the vertices are in $S$ and one is outside. Say $u \in S$ and $v \notin S$. Then
    $$1 = y_{\{u,v\}} = |0 - 1| = |x_u - x_v| .$$
  - If $\{u, v\} \notin \delta(S)$ then either $x_u = x_v = 0$ (both are in $S$) or $x_u = x_v$ (both are outside $S$). In either case $|x_u - x_v| = 0$ and so the constraint $y_{\{u,v\}} = 0 = |x_u - x_v|$ is again verified (with equality).
- $x_s = 0$ and $x_t = 1$. Moreover, we have $x_v \in \{0, 1\} \subseteq [0, 1]$ for every $v \in V$.

This finishes one part of the proof - there is an assignment to the variables such that the LP outputs $OPT$. This means that $OPT_{LP}$ is at most $OPT$, in other words $OPT_{LP} \leq OPT$.

Now let's prove that $OPT_{LP} \geq OPT$. Suppose that $(\{x_v^*\}_{v\in V}, \{y_e^*\}_{e\in E})$ is an optimal solution to the LP. Consider the following randomized rounding: select $\theta \in (0,1)$ uniformly at random and let $S = S_\theta = \{v \in V : x_v^* \leq \theta\}$. Let's analyze this rounding algorithm.

It is clear that we always output a feasible cut since $x_s^* = 0$ and $x_t^* = 1$. This tells us that for every $\theta \in (0,1)$ the associated $S_\theta$ is a valid solution and so $OPT \leq \delta(S_\theta)$. We thus have

$$OPT \leq \mathbb{E}_{\theta\in[0,1]}[|\delta(\{v : x_v^* \leq \theta\}|)].$$

We will now complete the proof by showing that the above expectation is at most $OPT_{LP}$. Let's introduce a new random variable $X_{e,\theta}$ that indicates if an edge is cut:

$$X_{e,\theta} = \begin{cases} 1 & \text{if } e \in \delta(S_\theta) \\ 0 & \text{otherwise.} \end{cases}$$

Then the expectation above equals

$$\mathbb{E}_{\theta\in[0,1]}\left[\sum_{e\in E} X_{e,\theta}\right] = \sum_{e\in E} \mathbb{E}_{\theta\in[0,1]}\left[X_{e,\theta}\right]$$

Let's analyze $\mathbb{E}_{\theta\in[0,1]}\left[X_e\right] = \Pr_{\theta\in[0,1]}[e \text{ is cut in } S_\theta]$ for a specific edge $e = \{u,v\} \in E$. In the case when $x_u^* \leq x_v^*$, the edge $e$ is cut if and only if $x_u^* \leq \theta \leq x_v^*$. The other case is analogous. It follows that

$$\Pr_{\theta\in[0,1]}[X_{\{u,v\},\theta}] = \begin{cases} \Pr_{\theta\in[0,1]}[\theta \in [x_u^*, x_v^*]] & \text{if } x_u^* \leq x_v^* \\ \Pr_{\theta\in[0,1]}[\theta \in [x_v^*, x_u^*]] & \text{if } x_u^* > x_v^* \end{cases} = \left|x_u^* - x_v^*\right|.$$

Now since the LP guarantees that $y_{\{u,v\}}^* \geq |x_u^* - x_v^*|$, we have

$$\sum_{\{u,v\}\in E} \mathbb{E}_{\theta\in[0,1]}\left[X_{\{u,v\},\theta}\right] = \sum_{\{u,v\}\in E} |x_u^* - x_v^*| \leq \sum_{\{u,v\}\in E} y_{\{u,v\}}^* = OPT_{LP}.$$

It follows that

$$OPT \leq \mathbb{E}_{\theta\in[0,1]}[|\delta(\{v : x_v^* \leq \theta\}|)] \leq OPT_{LP}$$

and this finishes the proof.