Exercise Set III

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 You have just started your prestigious and important job as the Swiss Cheese Minister. As it turns out, different fondues and raclettes have different nutritional values and different prices:

Food	Fondue moitie moitie	Fondue a la tomate	Raclette	Requirement per week
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin B [mg/kg]	60	300	0.5	15 mg
Vitamin C [mg/kg]	30	20	70	4 mg
[price [CHF/kg]	50	75	60	—

Formulate the problem of finding the cheapest combination of the different fondues (moitie moitie & a la tomate) and Raclette so as to satisfy the weekly nutritional requirement as a linear program.

2 Consider the following linear program for finding a maximum-weight matching:

$$\begin{aligned} \text{Maximize} \quad & \sum_{e \in E} x_e w_e \\ \text{Subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

(This is similar to the perfect matching problem seen in the lecture, except that we have inequality constraints instead of equality constraints.) Prove that, for bipartite graphs, any extreme point is integral.

3 *(half a *)* Use the integrality of the bipartite perfect matching polytope (as proved in class) to show the following classical result:

> The edge set of a k-regular bipartite graph $G = (A \cup B, E)$ can in polynomial time be partitioned into k disjoint perfect matchings.

A graph is k-regular if the degree of each vertex equals k. Two matchings are disjoint if they do not share any edges.

4 *(*)* Consider the linear programming relaxation for minimum-weight vertex cover:

$$\begin{aligned} \text{Minimize} \quad & \sum_{v \in V} x_v w(v) \\ \text{Subject to} \quad & x_u + x_v \geq 1 \quad \forall \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \qquad \forall v \in V \end{aligned}$$

In class, we saw that any extreme point is integral when considering bipartite graphs. For general graphs, this is not true, as can be seen by considering the graph consisting of a single triangle. However, we have the following statement for general graphs:

> Any extreme point x^* satisfies $x^*_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Prove the above statement.

5 *(*)* Consider an undirected graph $G = (V, E)$ and let $s \neq t \in V$. Recall that in the min s,t-cut problem, we wish to find a set $S \subseteq V$ such that $s \in S$, $t \notin S$ and the number of edges crossing the cut is minimized. Show that the optimal value of the following linear program equals the number of edges crossed by a min s,t-cut:

$$\begin{aligned} \textbf{minimize} \quad & \sum_{e \in E} y_e \\ \textbf{subject to} \quad & y_{\{u,v\}} \geq x_u - x_v \qquad \text{for every } \{u, v\} \in E \\ & y_{\{u,v\}} \geq x_v - x_u \qquad \text{for every } \{u, v\} \in E \\ & x_s = 0 \\ & x_t = 1 \\ & x_v \in [0, 1] \qquad \text{for every } v \in V \end{aligned}$$

The above linear program has a variable x_v for every vertex $v \in V$ and a variable y_e for every edge $e \in E$.

Hint: Show that the expected value of the following randomized rounding equals the value of the linear program. Select θ uniformly at random from $[0, 1]$ and output the cut $S = \{v \in V : x_v \leq \theta\}$.