Problem Set I

Solutions to many homework problems, including problems on this set, are available on the Internet or can be obtained by an LLM, either for exactly the same problem formulation or for some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Michael Kapralov. You are, however, allowed to discuss problems in groups of up to three students.

1 A new type of matroid. *(30pts)* Let $G = (V, E)$ be a directed graph, and let $S, D \subseteq V$ be sets of "starting" and "destination" vertices (not necessarily disjoint). We say that a subset I of D is *linked* to S if the vertices of I can be reached from S via vertex disjoint paths. We define $\mathcal{I}$ to be the set of all subsets of D which are linked to S. Prove that $\mathcal{I}$ is a matroid.

Hint. Let P_X be some set of paths which link W to X and let P_Y be some set of paths which link W to Y. Let $P_X(v)$ the path in P_X which ends in $v \in X$ and, correspondingly, let $P_Y(v)$ be the path in P_Y which ends in $v \in Y$. Let N be the number of *intersections* between paths in P_X and P_Y where by an "intersection" of paths p and q we mean the maximal consecutive sequence of vertices in p which is also a consecutive sequence of vertices in q. Note that this implies that two paths may have more than one intersection.

To prove the second axiom for matroids – that is, that a smaller independent set can be augmented with an element from a bigger independent set – it might be useful to show that it is possible to rebuild P_Y into a set of vertex-disjoint paths $P_{Y'}$ linking Y' to W so that $|Y'| = |Y|$, $Y' \setminus X \subseteq Y \setminus X$ and one of the following happens:

- The number of intersections of P_X and $P_{Y'}$ is bounded by $N - 1$,
- The new collection of endpoints, Y', satisfies $|X \setminus Y'| = |X \setminus Y| - 1$.

2 Intersection of 3 matroids. *(30pts)* We have seen in class that there exists an efficient algorithm for matroid intersection, i.e. a polynomial-time algorithm that given two matroids[1] over the same ground set, returns a common independent set of maximum size.

In this problem, you will show that finding an independent set of maximum size in the intersection of more than two matroids is unlikely to admit an efficient algorithm. More precisely, consider the following problem, called 3-MATROID-INTERSECTION:

- **input:** three matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2), \mathcal{M}_3 = (E, \mathcal{I}_3)$ over the ground set E, and an integer k;
- **ouput**: YES if there exists a set $S \subseteq E$ of size at least k that is independent in each of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ (i.e. $|S| \geq k$ and $S \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$), NO otherwise

You are asked to show that 3-MATROID-INTERSECTION is NP-hard.

Hint. In the DIRECTED-HAMILTONIAN-PATH problem, the input is a directed graph $G = (V, E)$, and one is asked to output YES or NO depending on whether G has a Hamiltonian path[2]. This problem is known to be NP-hard.

[1] With an efficient independence oracle.

[2] A Hamiltonian path is a directed simple path that visits every vertex exactly once.

3 Transportation. *(40pts)* In the minimum transportation problem, we are given a road network (of possibly one-way streets) and a central warehouse, and the goal is to select a cheap subset of roads that enables to send goods from the central warehouse to all other vertices respecting the direction of traffic. More formally, we consider the following problem:

- **input:** a directed graph $G = (V, E)$, a vertex $c \in V$ such that there is a directed path from c to every other vertex in G, and edge weights $w_e \in \mathbb{R}$ for every $e \in E$;
- **output:** a set of edges $F^* \subseteq E$ of minimum weight such that there is a directed path from c to every other vertex in the directed graph (V, F^*).

Consider the following linear programming relaxation of this problem[3].

$$\begin{aligned} &\min \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad &\sum_{e \in \delta^-(S)} x_e \geq 1, \quad \forall \emptyset \neq S \subsetneq V \setminus \{c\} \qquad \text{(LP)} \\ &x_e \geq 0, \quad \forall e \in E \end{aligned}$$

The goal of this problem is to show that the extreme points of the above linear program are integral. To do so, we will make use of the following fact.

Fact 1. *Any extreme point of a linear program can be written as the unique solution of a linear system of tight constraints[4]. In other words, consider a linear program over $y \in \mathbb{R}^n$ whose feasible region is given by $Ay \geq b$ where $A \in \mathbb{R}^{t \times n}, b \in \mathbb{R}^t$: then, for any extreme point y^* there is a subset $R \subseteq [t]$ of size n such that y^* is the unique solution of the linear system $A'y = b'$, where A' and b' are the restriction of A and b to the rows in R (so the rows of A' are linearly independent).*

Let $x \in \mathbb{R}^E$ be an extreme point solution to the above (LP) for the minimum transportation problem.

Assumption 2. *Without loss of generality, we assume that $x_e > 0$ for all $e \in E$. Indeed, if this was not the case, we could consider the graph G' obtained by only keeping the edges in the support of x, and the restriction of x to the edges in its support would be an extreme point of the linear program for G'.*

By virtue of this assumption, none of the constraints of the form $x_e \geq 0$ is satisfied with equality by x. Let $\mathcal{F}$ be the collection of vertex subsets such that $S \in \mathcal{F}$ if the corresponding constraint is tight for x, i.e. the constraint is satisfied with equality. In particular, the extreme point solution x is the unique solution to the linear system,

$$\sum_{e \in \delta^-(S)} x_e = 1, \qquad \forall S \in \mathcal{F}.$$

[3]Throughout this problem, for a set of vertices $S \subseteq V$, we use $\delta^-(S)$ to denote the subset of edges (u, v) of G going into S, i.e. such that $u \notin S, v \in S$.

[4]A constraint is said to be tight for a feasible solution if it is satisfied with equality.

As a first step, you will show that the tight constraints $\mathcal{F}$ are implied by a *laminar*[5] subset of tight constraints $\mathcal{L} \subseteq \mathcal{F}$. With $\mathbb{1}_F \in \{0,1\}^E$ we denote the characteristic vector of $F \subseteq E$, defined as

$$(\mathbb{1}_F)_e = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{if } e \notin F \end{cases} .$$

3a *(15 pts)* Show that if $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then the following three conditions hold:

- $S \cup T$, $S \cap T \in \mathcal{F}$;
- $E(S \setminus T, T \setminus S) = E(T \setminus S, S \setminus T) = \emptyset$;
- $\mathbb{1}_{\delta^-(S)} + \mathbb{1}_{\delta^-(T)} = \mathbb{1}_{\delta^-(S \cup T)} + \mathbb{1}_{\delta^-(S \cap T)}$.

3b *(15 pts)* Let $\mathcal{L} \subseteq \mathcal{F}$ be a maximal laminar subfamily[6] of $\mathcal{F}$. Show that

$$\text{span}\{\mathbb{1}_{\delta^-(S)} \mid S \in \mathcal{L}\} = \text{span}\{\mathbb{1}_{\delta^-(S)} \mid S \in \mathcal{F}\} .$$

Hint: Suppose that this is not true. Among all sets $T \in \mathcal{F}$ with $\mathbb{1}_{\delta^-(T)} \notin \text{span}\{\mathbb{1}_{\delta^-(S)} \mid S \in \mathcal{L}\}$ choose one that minimizes the number of sets in $\mathcal{L}$ for which T violates laminarity, i.e. pick $T \in \mathcal{F}$ as to minimize $|\{L \in \mathcal{L} \mid L \cap T \neq \emptyset, L \setminus T \neq \emptyset, T \setminus L \neq \emptyset\}|$. Use the result of the previous question to arrive at a contradiction.

From the result of **3a** and **3b**, we can conclude that x is the unique solution of a linear system

$$\sum_{e \in \delta^-(S)} x_e = 1, \qquad \forall S \in \mathcal{L} ,$$

where $\mathcal{L}$ is a laminar family of tight constraints. Next, you will show that x has to be integral. To do so, you can make use of the following theorem.

Theorem 3. *Let y be the unique solution to a linear system $Ay = b$ where A and b have entries in $\{0,1\}$. Then y is integral if for every subset R of rows of A, there exists a partition of R into two (possibly empty) parts R_1 and R_2 such that the vector*

$$\sum_{i \in R_1} A_i - \sum_{i \in R_2} A_i$$

has all entries in $\{-1, 0, 1\}$. Here A_i denotes the i-th row of A.

3c *(10 pts)* Use Theorem 3 to show that x is in fact integral.

[5] Remember that a collection of sets $\mathcal{L}$ is called laminar if for all $S, T \in \mathcal{L}$ either $S \subseteq T$, or $T \subseteq S$, or $S \cap T = \emptyset$.

[6] This means that adding any other set $S \in \mathcal{F}$ to $\mathcal{L}$ would violate its laminarity.