Graph Theory

Practice exam 1

Problem 1. Let T be a tree on $n \geq 2$ vertices with no vertices of degree two. Show that T has at least $\frac{n}{2} + 1$ leaves.

Problem 2. Let G be a planar graph with no cycles of length $3, 4, 5$. Show that G is 3-colorable.

Problem 3. Show that if every edge of a connected graph G belongs to an odd number of cycles then G has an Euler tour.

Problem 4. Let $n \cdot K_2$ denote the graph on $2n$ vertices consisting of n disjoint edges.

(a) Consider the following red/blue edge-colouring of K_{3n-2}:

  (i) partition the vertex set of K_{3n-2} into two sets A and B such that $|A| = 2n - 1$ and $|B| = n - 1$;

  (ii) colour any edge between two vertices of A with red;

  (iii) colour any edge touching a vertex of B with blue.

  Using this colouring, prove that $R(n \cdot K_2, n \cdot K_2) > 3n - 2$ for any $n \geq 1$.

(b) Prove that $R(n \cdot K_2, n \cdot K_2) \leq 3n - 1$ for any $n \geq 1$.

Problem 5. Show that if G is 3-connected then it contains vertices x_1, x_2, x_3, x_4 and internally vertex-disjoint paths $P_{i,j}$ for all $1 \leq i < j \leq 4$ such that $P_{i,j}$ has endpoints x_i and x_j.

Problem 6. Let H be a bipartite graph with classes A and B, such that $d(a) \geq 1$ for all $a \in A$, and $d(a) \geq d(b)$ for all $(a, b) \in E(H)$. Show that H contains a matching which covers every vertex in A.