Graph theory

Solutions to Practice exam 1

Problem 1. Let T be a tree on $n \geq 2$ vertices with no vertices of degree two. Show that T has at least $\frac{n}{2} + 1$ leaves.

Solution. A tree on n vertices has $n-1$ edges and by the handshaking lemma (Proposition 1.22), the sum of the degrees of all vertices equals $2(n-1)$. Let L denote the number of leaves. Since every non-leaf has degree at least 3, we have that

$$2(n-1) = \sum_{v \in V(T)} d(v) \geq L + 3(n-L).$$

Rearranging the inequality yields $L \geq \frac{n}{2} + 1$, as claimed.

Problem 2. Let G be a planar graph with no cycles of length $3, 4, 5$. Show that G is 3-colorable.

Solution. We will show that G is 2-degenerate, which by Theorem 7.19 implies that G is 3-colorable. Let G' be a subgraph of G, and let us show that it has a vertex of degree at most 2. We may assume that G' is connected by adding edges to it if needed (keeping it planar and not creating any new cycle). By Euler's formula (Theorem 6.10), its number of vertices n, edges e and faces f satisfy $n + f = e + 2$. On the other hand, since every edge is contained in at most two faces we have that $6f \leq 2e$, as every face is of length at least 6 (this is also true for the outer face if G' has at least 3 edges). Plugging this into Euler's formula, we have $n + e/3 \geq e + 2$, which after transforming gives $2e \leq 3n - 6$, which means that the average degree is $\frac{2e}{n} < 3$, so G' contains a vertex of degree at most 2, finishing the proof.

Problem 3. Show that if every edge of a connected graph G belongs to an odd number of cycles then G has an Euler tour.

Solution. It is enough to show that every vertex of G has even degree because a connected graph with even degrees has an Euler tour (by Theorem 4.5). Assume that there is a vertex v which has an odd number of neighbours, say it has $2k-1$ neighbours. We double count the number N of cycles passing through v. Let d_i be the number of cycles that i-th edge incident to v belongs to. Then $2N = d_1 + \ldots + d_{2k-1}$ since every cycle through v contains exactly 2 edges incident to v. But we know that each d_i is odd by assumption and there is an odd number of them, so the sum $d_1 + \cdots + d_{2k-1}$ must be odd, a contradiction.

Problem 4. Let $n \cdot K_2$ denote the graph on $2n$ vertices consisting of n disjoint edges.

(a) Consider the following red/blue edge-colouring of K_{3n-2}:

  (i) partition the vertex set of K_{3n-2} into two sets A and B such that $|A| = 2n - 1$ and $|B| = n - 1$;

  (ii) colour any edge between two vertices of A with red;

  (iii) colour any edge touching a vertex of B with blue.

  Using this colouring, prove that $R(n \cdot K_2, n \cdot K_2) > 3n - 2$ for any $n \geq 1$.

(b) Prove that $R(n \cdot K_2, n \cdot K_2) \leq 3n - 1$ for any $n \geq 1$.

Solution. **(a)** We prove that the colouring defined in the question does not contain a red $n \cdot K_2$ or a blue $n \cdot K_2$. Indeed, any vertex in a red $n \cdot K_2$ must have a red edge incident to it, so any such vertex must be in A. But there are only $2n - 1$ vertices in A, so there is no red $n \cdot K_2$. Moreover, any blue edge must have a vertex in B, but there are only $n - 1$ vertices in B, so there is no blue $n \cdot K_2$ either.

(b) We prove the statement by induction on n. The case $n = 1$ is clear. Now assume that $n > 1$. Consider a red/blue edge-colouring of K_{3n-1}. If every edge in the colouring is red, then there exists a red $n \cdot K_2$ since $3n - 1 \geq 2n$. Similarly, if every edge in blue, then there exists a blue $n \cdot K_2$. Else, there exist a red edge and a blue edge as well. It is not hard to see that then there must exist a red and a blue edge sharing a vertex (else, we would have a partition of the vertex set into a set X of vertices incident with only red edges and a set Y of vertices incident with only blue edges, but then the edges between X and Y cannot get any colour). Let uv be a red edge and let uw be a blue edge. Consider the colouring induced by the removal of vertices u, v and w. This is a red/blue colouring of $K_{3(n-1)-1}$, so by the induction hypothesis, it contains a red $(n-1) \cdot K_2$ or a blue $(n-1) \cdot K_2$. In the former case, we can add the (red) edge uv to obtain a red $n \cdot K_2$, while in the latter case we can add the (blue) edge uw to obtain a blue $n \cdot K_2$.

Problem 5. Show that if G is 3-connected then it contains vertices x_1, x_2, x_3, x_4 and internally vertex-disjoint paths $P_{i,j}$ for all $1 \leq i < j \leq 4$ such that $P_{i,j}$ has endpoints x_i and x_j.

Solution. Let C be a shortest cycle in G. Then C has no chords. Since G is 3-connected, it cannot be that $V(G) = V(C)$. So let $x \in V(G) \setminus V(C)$. Since G is 3-connected, the minimum number of vertices distinct from x separating x from C is at least 3. Hence, by the Fan Lemma (Corollary 3.15), there are three paths Q_1, Q_2, Q_3 from x to $V(C)$ which intersect each other only at x and intersect $V(C)$ only in their endpoints. Let $y_i \in V(C)$ be the endpoint of Q_i different from x. Then the vertices x, y_1, y_2, y_3 and the paths $Q_1, Q_2, Q_3, C[y_1, y_2], C[y_2, y_3], C[y_3, y_1]$ satisfy the conditions of the question (here, $C[u, v]$ denotes the segment of the cycle C between u and v).

Problem 6. Let H be a bipartite graph with classes A and B, such that $d(a) \geq 1$ for all $a \in A$, and $d(a) \geq d(b)$ for all $(a, b) \in E(H)$. Show that H contains a matching which covers every vertex in A.

Solution. For any set $S \subset A$, let $N(S)$ be the set of vertices in B which have at least one neighbour in S. If $|N(S)| \geq |S|$ holds for each $S \subset A$, then Hall's theorem (Theorem 5.7) implies that H contains a matching which covers every vertex in A. Else, let T be a minimal subset of A for which $|N(T)| < |T|$. Let R be an arbitrary subset of T of size $|T| - 1$. Then, by the minimality assumption for T, we have $|N(R)| = |R|$ and for any $S \subset R$, we have $|N(S)| \geq |S|$. Hence, by Hall's theorem, H contains a matching f which covers every vertex in R. By the condition that $d(a) \geq d(b)$ for all $(a, b) \in E(H)$, we have

$$\sum_{a \in R} d(a) \geq \sum_{b \in f(R)} d(b) = \sum_{b \in N(R)} d(b),$$

so

$$\sum_{a \in T} d(a) > \sum_{b \in N(R)} d(b).$$

Moreover, since $|N(T)| < |T|$ and $|N(R)| \geq |R| = |T| - 1$, we have $N(T) = N(R)$. But $\sum_{a \in T} d(a)$ is equal to the number of edges between T and $N(T)$, while $\sum_{b \in N(R)} d(b) = \sum_{b \in N(T)} d(b)$ is an upper bound for this number, so $\sum_{a \in T} d(a) \leq \sum_{b \in N(R)} d(b)$, which is a contradiction.