Graph theory

Solutions to Practice exam 2

Problem 1. Let G be a bipartite graph with parts $A = \{a_1, \dots, a_n\}$ and B. Suppose that $d(a_i) = i$ for every $1 \leq i \leq n$. Show that G has a matching covering A.

Solution. A matching covering A can be constructed greedily: for $i = 1, \dots, n$ in increasing order, choose a neighbour of a_i which was not chosen by any a_j with $j < i$. This is possible because $d(a_i) = i$.

Problem 2. Let $x_1, \dots, x_n$ be irrational numbers. Show that there are at most $\lfloor \frac{n^2}{4} \rfloor$ pairs $1 \leq i < j \leq n$ such that $x_i + x_j \in \mathbb{Q}$.

Solution. Let G be a graph whose vertex set is $\{x_1, \dots, x_n\}$, and in which $x_i x_j$ is an edge if and only if $x_i + x_j \in \mathbb{Q}$. Suppose for contradiction that G has more than $\lfloor n^2/4 \rfloor$ edges. Since the n-vertex 2-partite Turán graph has $\lfloor n^2/4 \rfloor$ edges, G contains a triangle x_i, x_j, x_k by Turán's theorem (Theorem 9.5). By definition, $(x_i + x_k), (x_i + x_j), (x_j + x_k) \in \mathbb{Q}$, so the sum of these three rational numbers $2(x_i + x_j + x_k)$ is also in $\mathbb{Q}$, implying that $x_i + x_j + x_k \in \mathbb{Q}$. Finally, since $x_i + x_j \in \mathbb{Q}$ and $x_k \notin \mathbb{Q}$, we have that their sum $x_i + x_j + x_k \notin \mathbb{Q}$, a contradiction.

Problem 3. Let G be a graph on n vertices with at least $2n - 2$ edges. Show that G contains two cycles of the same length.

Solution. First we claim that G has at least $n - 1$ different cycles. Indeed, we repeatedly find a cycle in G and delete one of its edges. We can continue this as long as the graph has at least n edges (because a graph with n vertices and at least n edges has a cycle). Hence, we can continue this for $n - 1$ rounds, giving $n - 1$ different cycles.

The length of a cycle is between 3 and n. Therefore, there are $n - 2$ possible lengths. By the pigeonhole principle, two of the $n - 1$ cycles have the same length.

Problem 4. Let G be a graph with $\chi(G) = k$.

(a) Show that $e(G) \geq \binom{k}{2}$.

(b) Suppose further that $e(G) = \binom{k}{2}$ and G has no isolated vertices. Show that $G = K_k$.

Solution. **(a)** Let G be a graph with $\chi(G) = k$ and consider a proper coloring of G with k colors. For $i \in [k]$, let V_i denote the set of vertices with color i. Suppose that $e(G) < \binom{k}{2}$. Then, for some pair (i, j) there are no edges between V_i and V_j in G. However, then we can recolor all vertices in V_j with color i. This remains a proper coloring, but uses $k - 1 < \chi(G)$ colors, a contradiction. So, $e(G) \geq \binom{k}{2}$.

(b) Assume first that there is a vertex $v \in V(G)$ with $d(v) \leq k-2$. Consider $G' = G[V(G) \setminus \{v\}]$. If $\chi(G') \geq k$, then by part (a), we have $e(G') \geq \binom{k}{2}$ and $e(G) = d(v)+e(G') > \binom{k}{2}$ since G has no isolated vertices, a contradiction. Thus, $\chi(G') \leq k - 1$. Consider a proper $(k-1)$-coloring of G'. Since $d(v) \leq k - 2$, one of the $k - 1$ colors was not used on the neighbours of v, so we can extend this coloring into a proper $(k - 1)$-coloring of G, a contradiction. Therefore $d(v) \geq k - 1$ for every $v \in V(G)$. Then $2e(G) \geq |V(G)|(k - 1)$. On the other hand, $2e(G) = k(k - 1)$, so $|V(G)| \leq k$. Since $\chi(G) = k$, it follows that $G = K_k$.

Problem 5. Let G be a $(k+1)$-connected graph, and let $a, b, x_1, \ldots, x_k$ be distinct vertices in G. Show that there is a path from a to b containing all vertices $x_1, \ldots, x_k$.

Solution. Consider a path P from a to b in G that contains the maximum possible number of vertices of the set $X = \{x_1, \ldots, x_k\}$, and suppose for the purpose of contradiction that there is a vertex x_i not on this path. Note that we must delete at least $\min\{|P|, k + 1\}$ vertices (distinct from x_i) to separate x_i from $V(P)$, so by Corollary 3.15 (the "fan lemma") there is an x_i-$V(P)$ fan F with at least $\min\{|P|, k+1\}$ paths. The vertices of X split P into at most k edge-disjoint subpaths, each starting and ending in either a, b or a member of X (this is because, by assumption, P contains at most $k - 1$ elements of X). If $|P| \geq k + 1$ then $\min(|P|, k + 1) = k + 1$, so two paths of F end on the same subpath of P. Then we can add x_i to the path by using these two paths of F, yielding a new path from a to b that contains more elements of X than P did, contradicting our choice of P. Similarly if $|P| \leq k$ then F contains a path towards any vertex of P so in particular we can add x_i by using the paths towards a and its neighbour on P. In either case we obtain a contradiction.

Problem 6. Let G be a graph on $n \geq 6$ vertices with minimum degree at least $n/2$. Prove that there exist two vertex-disjoint cycles in G which together cover the vertex set of G. For the purpose of this problem, we consider a single vertex and a single edge to be cycles.

Solution. Dirac's theorem (Theorem 4.13) tells us that there is a Hamilton cycle $(v_1 v_2 \ldots v_n)$. Notice that if there is an edge $v_i v_{i+2}$ (all the indices are taken modulo n) then v_{i+1} and $(v_i v_{i+2} v_{i+3} \ldots v_{i-1})$ give the desired cycles. Similarly, if there is an edge of the form $v_i v_{i+3}$, then $v_{i+1} v_{i+2}$ and $(v_i v_{i+3} v_{i+4} \ldots v_{i-1})$ are cycles of the desired form. Hence, we may assume that all but 2 neighbours of v_1 are among $v_5, \dots, v_{n-3}$ and that all but 2 neighbours of v_n are among $v_4, \dots, v_{n-4}$. Notice further that if both $v_1 v_i$ and $v_n v_{i+1}$ are edges, for some $3 \leq i \leq n-3$, then $(v_1 \dots v_i)$ and $(v_{i+1} \dots v_n)$ give us the desired cycles. So let $S := N(v_1) \setminus \{v_n, v_2\} \subseteq \{v_5, \dots, v_{n-3}\}$, $T := N(v_n) \setminus \{v_1, v_{n-1}\} \subseteq \{v_4, \dots, v_{n-4}\}$ and $S^+ := \{v_{i+1} \mid v_i \in S\} \subseteq \{v_6, \dots, v_{n-2}\}$. If $S^+ \cap T \neq \emptyset$, then by the above observation we are done. Notice that $S^+, T \subseteq \{v_4, \dots, v_{n-2}\}$ and hence $|S^+ \cup T| \leq n-5$, but $|S^+| + |T| = |S| + |T| \geq 2 \cdot (n/2 - 2) = n - 4$, so indeed $S^+ \cap T$ is non-empty.