Graph Theory

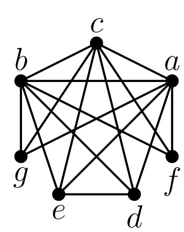
Solutions 1

Problem 1(a): The sum of degrees $3+3+2+2+2+1$ is odd, so by the handshaking lemma, there is no such a graph.

Problem 1(b): There are 7 vertices in total, so any vertex of degree 6 must be adjacent to all other vertices. There are 3 vertices of degree 6, so every vertex must have degree at least 3. Hence, no such graph exists.

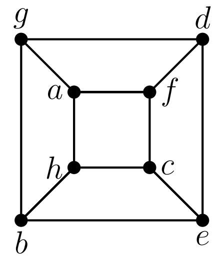
Problem 1(c): Let us call the vertices a, b, c, d, e, f, g, h and suppose they are ordered by their degrees (so $d(a), d(b), d(c), d(d) = 6$, $d(e) = 5$, $d(f) = 4$, $d(g) = 2$, $d(h) = 1$). We have 8 vertices, so each one of degree 6 is connected to all but one of the other vertices. So each of a, b, c, d sends at least one edge to the set $\{g, h\}$, but that is impossible, since g and h are adjacent to at most 3 edges in total. Hence there is no such graph.

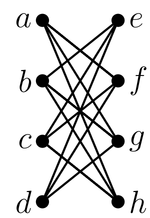
Problem 1(d):



Explanation: Let us denote the vertices by a, b, c, d, e, f, g, where a, b, c have degree 6, d, e have degree 4 and f, g have degree 2. There are 7 vertices in total, so a, b, c must be adjacent to all other vertices. This already gives 3 edges incident to each of d, e, f, g. So it only remains to increase the degree of d, e by 1. This is done by adding an edge between d, e.

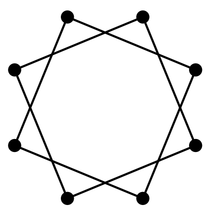
Problem 2(a): The first two graphs are isomorphic. Here are vertex labelling which show it:

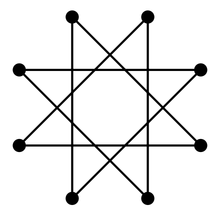




The last graph is not isomorphic to the other two. For example, it has a cycle of length 5 while the other two do not.

Problem 2(b): It is sometimes convenient to look at the complements of graphs. In this case, the complements of the two graphs are:





So the complement of the graph on the right is a cycle of length 8, while the complement of the graph on the left is the disjoint union of two cycles of length 4. Hence, the graphs are not isomorphic.

Problem 3: Let $C_1, \dots, C_t$ be the connected components of G, $t \geq 2$. In $\bar{G}$, every two vertices in different C_i's are adjacent. So trivially, there is a path in $\bar{G}$ between any two vertices in different C_i's. There is also a path between vertices in the same C_i by going through some C_j with $j \neq i$. This shows that $\bar{G}$ is connected.

The converse is not true: it may be that both G and $\bar{G}$ are connected; for example, if G is a path of with four vertices.

Problem 4: Let G be a graph on $n \geq 2$ vertices. Note that the degree of a vertex in G lies in $\{0, 1, 2, \dots, n-1\}$. Since there are n vertices and n possible values for their degrees, if they are all distinct then there must be exactly one vertex of each possible degree. In particular, there must be a vertex v of degree 0 (i.e. v is not adjacent to any other vertex) and a vertex u of degree $n-1$ (i.e. u is adjacent to any other vertex). However, these two things cannot both happen and therefore we have a contradiction. We conclude that there are two vertices in G of equal degree.

Problem 5: We do induction on n. If $n = 7$ then we have $35 - 14 = 21$ edges. The only graph on 7 vertices with 21 edges is K_7, so we can take K_7 as a subgraph satisfying the conditions.

Now suppose that $n > 7$ and we know the statement for $n - 1$. If all the degrees in the graph are at least 6, we can again take the graph itself as the desired subgraph. If not, then there is a vertex v with $d(v) \leq 5$. Then $G - v$ is a graph on $n - 1$ vertices, and it has at least $5n - 14 - 5 = 5(n - 1) - 14$ edges (we lose at most 5 by deleting v). So by induction $G - v$ has a subgraph with minimum degree at least 6, but that is also a subgraph of G with minimum degree at least 6.

Problem 6: Suppose there are two vertex-disjoint paths P_1 and P_2 of maximum length l. Since the graph is connected, there is a path Q between P_1 and P_2, say from $w_1 \in P_1$ to $w_2 \in P_2$ such that the interior vertices of Q avoid P_1 and P_2 (or formally: $V(Q) \cap (V(P_1) \cup V(P_2)) = \{w_1, w_2\}$). Here w_1 and w_2 cut P_1 and P_2 into two pieces each. Let P_1' be the longer piece of P_1 and P_2' be the longer piece of P_2, so both P_1' and P_2' have length at least $l/2$. Moreover, $P_1' \cup Q \cup P_2'$ forms a path(!) of length at least $l + e(Q)$, which is impossible because the longest path had length l. This is a contradiction.

To say that $P_1' \cup Q \cup P_2'$ is a path, we really needed the fact that Q is internally disjoint from P_1 and P_2. But why is there such a Q? To show this, take arbitrary vertices $v_1 \in P_1$ and $v_2 \in P_2$. As G is connected, there is a path Q_0 from v_1 to v_2. Since P_1 and P_2 are vertex-disjoint, there is a *last* vertex in Q_0 from P_1, let us call this w_1. We define w_2 to be the first vertex from P_2 appearing after w_1 in Q_0. Then the part of the path Q_0 between w_1 and w_2 is a good choice for Q.

Problem 7: Here we will only sketch a proof that if G has no odd cycles then it is bipartite.

First, note that if the connected components of G are bipartite then so is G. Therefore, we may assume without loss of generality that G is connected (if not, apply the following reasoning to its connected components).

Now, let $v \in V(G)$ be any vertex. Define the sets

$$X = \{u \in V(G) : \text{there is a path of } \textbf{odd} \text{ length from } v \text{ to } u\}$$

and

$$Y = \{u \in V(G) : \text{there is a path of } \textbf{even} \text{ length from } v \text{ to } u\}.$$

We claim that these sets form a bipartition of G, i.e., $X \cup Y = V(G)$, $X \cap Y = \emptyset$ and there are no edges inside X and no edges inside Y. Since G is connected, it is clear that $X \cup Y = V(G)$.

Suppose by contradiction that $X \cap Y \neq \emptyset$ and let u be a vertex in $X \cap Y$. Since $u \in X$, there exists a path P_1 of odd length from v to u. Furthermore, since $u \in Y$, there exists a path

P_2 of even length from v to u. In particular, we can obtain a **closed walk** of odd length (not necessarily a cycle!) by starting in v, going through P_1 to u and coming back to v through P_2. Therefore, if we can show that **any closed walk of odd length contains a cycle of odd length**, then we would obtain a contradiction with the initial assumption that G has no odd cycle.

Similarly, suppose there is an edge $e = u_1u_2$ between two vertices u_1, u_2 in the same part, say, in part X. Because these vertices lie in X, there must be paths Q_1 and Q_2 of odd length from v to u_1 and u_2, respectively. Note that we can obtain a **closed walk** of odd length by starting at v, going through Q_1 to u_1, traversing e to u_2 and coming back to v through Q_1. Therefore, if we can show that **any closed walk of odd length contains a cycle of odd length**, then we would obtain a contradiction with the initial assumption that G has no odd cycle.

By the previous two paragraphs, we see that in order to show that (X, Y) is a bipartition of G it suffices to prove that any closed walk of odd length contains a cycle of odd length. Suppose $W = v_1v_2 \dots v_kv_1$ is a closed walk of odd length. If W does not repeat vertices then W is a cycle of odd length. Otherwise, let $i \in \{1, 2, \dots, k\}$ be the smallest index for which there exists an index $j < i$ such that $v_j = v_i$ (i.e. if we walk along W the first vertex we repeat will be $v_j = v_i$). Note that $C = v_jv_{j+1} \dots v_i$ is a closed walk with no repeated vertices and therefore it is a cycle. If C is a cycle of odd length then we are done. Otherwise, C is a cycle of even length and therefore the walk $W_1 = v_1v_2 \dots v_{j-1}v_jv_{i+1}v_{i+2} \dots v_kv_1$ has odd length and is smaller than W (as W_1 is obtained from W by deleting C). Now, we can successively repeat the procedure above to W_1 to either find a cycle of odd length or to obtain closed walks $W_2, W_3, \dots$ of odd length such that $|W_1| > |W_2| > |W_3| > \dots$. Since the sizes of these walks are strictly decreasing this procedure must eventually terminate by finding a cycle of odd length in W.