Graph Theory

Solutions 2

Problem 1: Let the number of vertices of the tree be n. Then the number of edges is $n-1$. Since $n-1$ is even, n is odd. Since the sum of an odd number of odd numbers is odd, and the sum of degrees must be even (by the handshaking lemma), there must be a vertex of even degree.

Problem 2: Take a spanning tree T of G. It has at least two leaves, say x and y. Then $T-x$ and $T-y$ are both connected, hence so are their supergraphs, $G-x$ and $G-y$.

Problem 3: We prove the assertion by induction on k. To this end, it will be convenient to prove the assertion more generally for forests (and not just trees). So let F be a forest with precisely $2k$ vertices of odd degree. If $k=0$ then F has no edges (since any forest with at least one edge has at least two leaves, which have odd degree), and indeed the edge-set of F trivially decomposes into 0 paths (since it's empty).

Suppose now that $k \geq 1$. Take a nontrivial connected component of F. This component must have at least two leaves, so take such leaves x, y. Let $P = x, v_1, \ldots, v_\ell, y$ be the path in F between x and y. Consider the forest F' obtained from F by deleting the **edges** of P.

Observe that by doing this we changed the degrees of x and y by 1, so they become even, and the degrees of $v_1, \ldots, v_\ell$ by 2, so their parity does not change. The degree of any other vertex was not changed. Hence, F' has precisely $2k-2$ vertices of odd degree. By the induction hypothesis, the edge-set of F' decomposes into $k-1$ paths. Together with P, we get a decomposition of $E(F)$ into k paths, as required.

Problem 4: Here we only show that if G is a tree then any pairwise intersecting paths $P_1, \ldots, P_k$ intersect at a common vertex. This is trivial for $k \leq 2$, so let us start by showing it for $k=3$. Suppose P_1, P_2 and P_3 are pairwise intersecting but they have no common vertex. Look at $P_1 = v_1 v_2 \cdots v_l$ and color a vertex v_i red if it is also in P_2, color it blue if it is also in P_3 and leave it uncolored if it is only in P_1. By our conditions, no vertex is colored both red and blue, but there *is* some red vertex v_i and some blue vertex v_j.

We may assume that $i < j$. Then there is some vertex $v_{i'}$ with $i \leq i' < j$ such that $v_{i'}$ is red but $v_{i'+1}$ is not. Then the edge $e = v_{i'} v_{i'+1}$ is neither in P_2, nor in P_3. Now delete e from the tree, it splits the graph into two components. Each of P_2 and P_3 can only be in one of the components, because they do not contain e. But $v_i \in P_2$ and $v_j \in P_3$ are in different components, hence so are P_2 and P_3. Thus they cannot intersect, contradiction.

We could use the same idea to prove the statement for $k > 3$, but let us follow the standard way of proving Helly-type theorems, and use induction instead.

So suppose the statement is true for $k-1$. Then $P_2, \ldots, P_k$ share a vertex: u, and $P_1, P_3, \ldots, P_k$ also share a vertex: v. But then the unique u-v path in the tree – let us call it P' – is contained in all of $P_3, \ldots, P_k$. So if we can find a vertex shared by P_1, P_2 and P', it will be a good choice for all the paths. Now we can apply the $k = 3$ case to these three paths (the condition clearly holds) to find this common vertex.

Remark. There are many other ways to prove this result. Induction on the number of vertices gives a somewhat easier proof than the one above. Alternatively, one can also use the fact that the intersection of two paths is a path (but a rigorous proof of this and its application to the problem needs care).