Graph Theory

Solutions 4

Problem 1: Let X be a set separating u from v. This means that u and v are in different components of $G - X$ – let us call these components C_u and C_v, respectively. Now let us look at a vertex $x \in X$. If x is connected to both C_u and C_v in G, then adding it back to $G - X$ connects the components of u and v, or in other words: $X - x$ is not separating. On the other hand, if x is not connected to both, say it has no neighbor in C_v, then adding it back to $G - X$ cannot join C_u and C_v (in fact, it cannot affect C_v at all). So in this case $X - x$ is still separating.

This means that removing a vertex x from a separating set makes the set no longer separating iff x had neighbors in both C_u and C_v. Hence X is minimal separating iff all its vertices have neighbors in both C_u and C_v.

Problem 2: Let X be a set of at most $k-1$ vertices. We need to show that $G-X$ is connected. In fact, we are going to show something stronger: $G - X$ has diameter 2. So take two vertices u and v in $G - X$, we want to find a path between them. If they are adjacent vertices then we are done. If not, then note that both of them are adjacent to at least $\frac{n+k-2}{2}$ other vertices in G, and since there are $n - 2$ other vertices in total, these neighborhoods overlap in at least $\frac{n+k-2}{2} + \frac{n+k-2}{2} - (n - 2) = k$ vertices. Now X contains at most $k - 1$ of these k vertices, so there is at least one common neighbor left in $G - X$, providing the path between u and v that we were looking for.