Graph Theory

Solutions 5

Problem 1: Prove that a graph G with at least 3 vertices is 2-connected if and only if for any three vertices x, y, z there is a path from x to z containing y.

Solution: From Corollary 3.15, it follows that if G is k-connected, v is a vertex and S is a vertex set not containing v, then G contains a v-S fan of $\min(k, |S|)$ paths. Let us apply this to our 2-connected graph with $k = 2$, $v = y$ and $S = \{x, z\}$, then we obtain internally vertex disjoint paths from y to both x and z. By joining these two paths at y, we get a path from x to z through y.

For the other direction, suppose G has such a path for all x, y, z, but it is not 2-connected. If G is not connected then let x and z be in different connected components and let y be any other vertex (here we use that G has at least 3 vertices); then there can't be a path from x to z (going through y), contradicting our initial assumption. If G is connected but not 2-connected then, since G has at least 3 vertices, there is some cut vertex x that separates a vertex y from some other vertex z. By the assumption, there is an x-z path P containing y. Now delete the vertex x from G. $P - x$ is still a path in $G - x$. In particular, $G - x$ contains a y-z path (a subpath of $P - x$). But this contradicts our assumption about x separating y from z.

Problem 2: Let G be a k-connected graph, where $k \geq 2$. Show that if $|V(G)| \geq 2k$ then G contains a cycle of length at least $2k$.

Solution: Let G be a k-connected graph with $|V(G)| \geq 2k$, where $k \geq 2$. Let C be a cycle in G of maximal length and suppose, for the sake of contradiction, that $|C| \leq 2k - 1$. Since $|V(G)| \geq 2k$, there exists $v \in V(G) \setminus V(C)$. As mentioned above, the $2k$-connectedness of G implies that there are $t = \min(k, |C|)$ paths $P_1, \ldots, P_t$, each starting at v and ending at a vertex of C, such that $V(P_i) \cap V(P_j) = \{v\}$ for all $1 \leq i \neq j \leq t$. For each $1 \leq i \leq t$, let $x_i \in V(C)$ be the other endpoint of P_i. Since $|C| \leq 2k - 1$, we have $t > |C|/2$. Hence, there exist $i \neq j$ such that x_i, x_j are consecutive on the cycle C. Now, we can obtain a longer cycle by replacing the edge $\{x_i, x_j\} \in E(C)$ with the path x_i, P_i, v, P_j, x_j (this cycle is indeed longer because both P_i and P_j have at least one edge). We got a contradiction the maximality of C.

Problem 3: A matching is a set of pairwise-disjoint edges. Let G be a bipartite graph, and suppose that G has no matching of size k. Prove that there is a set $X \subseteq V(G)$, $|X| \leq k - 1$, such that X intersects every edge of G. This statement is called König's theorem.

Solution: Apply Menger's theorem with S, T being the two sides of a bipartition of G. If there are k vertex-disjoint S,T-paths, then, taking one edge from each path, we obtain a matching of size k, a contradiction. Hence, by Menger's, theorem, there is a set X of size at most $k-1$ which separates S,T. Since every edge of G goes between S and T, it follows that in $G-X$ there are no edges, so every edge intersects X.

Problem 4: Give a complete proof of Corollary 3.18 (ii) from the notes. That is, show that for every graph G and distinct vertices u,v, the minimum number of edges separating u from v in G is equal to the maximum number of edge-disjoint u-v-paths in G.

Solution: If there are k edge-disjoint u-v paths in G, then clearly at least k edges from G need to be removed to disconnect u from v. Thus, it remains to show the other direction, if one needs to remove k edges to disconnect u from v, then there are k edge-disjoint u-v paths in G.

As indicated, we shall consider the line graph $L(G)$ and apply Menger's theorem with S the set of edges incident to u and T the set of edges incident to v. Note that S,T are sets of vertices in $L(G)$. Thus, by Menger's theorem, the maximum number of vertex-disjoint S-T paths in $L(G)$ equals the minimum size of an S-T separating set in $L(G)$. Let us denote this number by k.

The rest of the proof relies on the following claim.

Claim: Let $F \subseteq E(G)$. Then, there is a u-v path in G using only edges in F if and only if there is a path from S to T in $L(G)$ using only vertices in $F \subseteq V(L(G))$.

Before proving the claim, let us see how to use it to finish the proof.

By the above, there are k vertex-disjoint S-T paths in $L(G)$. Let $F_1,\dots,F_k \subseteq V(L(G))$ denote the vertices of these paths. By the claim, there is a u-v path P_i in G using only edges in F_i. Since $F_1,\dots,F_k$ are disjoint, these paths are edge-disjoint, proving that there are k edge-disjoint u-v paths in G.

On the other hand, let $F^* \subseteq E(G)$ be an arbitrary set of edges. Letting $F = E(G) \setminus F^*$, by the claim, we get that there is u-v path in G using only edges in F if and only if there is a path from S and T in $L(G)$ using only F (as vertices in $L(G)$). Equivalently, removing F^* (as edges) disconnects u from v in G if and only if removing F^* (as vertices) disconnects S from T in $L(G)$. Recalling our application of Menger's theorem, we see that the minimum number of edges needed to remove to disconnect u from v equals k, finishing the proof.

Proof of the claim: Assume first there is a path from u to v using only edges in F. This path forms a path in $L(G)$ from S to T using only F when viewed as vertices of $L(G)$, proving one direction.

Now, assume there is a path $P = f_1,\dots,f_\ell$ from S to T in $L(G)$ using only F when viewed as vertices of $L(G)$. Recall that $f_1,\dots,f_\ell$ are edges in $E(G)$. Consider the graph G' with $V(G') = V(G)$ and $E(G') = F$. Our task is to show that u and v lie in the same connected component of G'. We shall prove by induction on i that all vertices in $S_i \coloneqq V(f_1) \cup V(f_2) \cup \ldots, V(f_i)$ lie in the same connected component of G'. (In words, S_i is the set of vertices that appear on the edges $f_1, \ldots, f_i$). For $i = 1$, the vertices in S_1 are the two endpoints of f_1 so they are connected in G'. For $i > 1$, the two edges f_{i-1}, f_i share a vertex by the definition of a line graph, so we may write $f_{i-1} = xy$ and $f_i = yz$, thus the three vertices x, y, z are in the same connected component of G', finishing the proof of the induction step. By definition, f_1 is an edge incident to u and f_ℓ an edge incident to v, thus $\{u, v\} \subseteq S_\ell$. We conclude that u and v are connected in G', as claimed.