Graph Theory

Solutions 6

Problem 1: Show that if $k > 0$ then the edge set of any connected graph with $2k$ vertices of odd degree can be split into k trails.

Solution: Pair up the odd-degree vertices of the graph and add an edge between any pair. To distinguish them, let us say that the new edges are blue. The resulting *multi*graph is connected with all degrees even, so it has an Euler tour. Now delete the blue edges from this tour: we get k trails that cover each edge of the original graph exactly once.

Alternative solution: We proceed by induction on k. For $k = 1$ the graph has an Euler trail as was shown in class. Now assume the statement is true for $1, \ldots, k-1$ and that the connected graph G has $2k$ odd degrees. Let u and v be two odd-degree vertices and P be a path connecting them. Then if we remove P from G, we get rid of 2 odd degrees. So we want to apply induction on $G - P$, but it might not be connected. Never mind that, we can apply induction on each of the components of $G - P$. Or can we?

Each component will have an even number of odd-degree vertices, say the i'th component has $2k_i$, and here $\sum 2k_i = 2k - 2$. By induction we can split the i'th component into k_i trails... **except** if $k_i = 0$! But then it has an Euler tour. Note that P must touch each new component, so we can extend P into a trail using this Euler tour. We do this for all such i and we apply induction on each component with $k_i > 0$.

Remark. Induction is powerful, it is not a bad idea to automatically try to apply induction as soon as it seems the inductive assumption would give some information about the problem (of course checking the basis is a good idea as well). However, there are often clever tricks such as the one in the intended solution which give shorter solutions and if induction fails are the only option.

Problem 2: Let G be a connected graph that has an Euler tour. Prove or disprove the following statements.

(a) If G is bipartite then it has an even number of edges.

(b) If G has an even number of vertices then it has an even number of edges.

(c) For edges e and f sharing a vertex, G has an Euler tour in which e and f appear consecutively.

Solution: a): Yes. Let X and Y make the bipartition of the vertex set. Then the number of edges in G is the sum of degrees of vertices in X (this is the so-called bipartite handshake lemma, it is good to keep in mind!). Since G has an Euler tour all degrees are even so indeed $|E(G)| = \sum_{v\in X} d(v)$ is even as each of $d(v)$ is even.

b): The answer is no but let us describe the thought process of finding the example. Being Eulerian tells us all degrees in our graph should be even. By handshake lemma we know that sum of degrees is equal to twice the number of edges, so we can divide by 2 to get $|E(G)| = \sum_{v\in V(G)} \frac{d(v)}{2}$ where each $d(v)/2$ is an integer. Our goal is to have an odd number of edges. This is equivalent with an odd number of v having $d(v)/2$ odd. Also since G is connected we know that $d(v) > 0$. Now we may try to guess what kind of degree sequence could possibly work. For example taking $d(v)/2 = 1$ for one vertex and all other $d(w)/2$ being even. This in particular means that all other vertices should have degree at least 4. This means there are at least 5 vertices but since we want an even number this leads us to at least 6 vertices. Now is it possible to have a graph with degree sequence $4, 4, 4, 4, 4, 2$? The answer is yes take a K_5 minus an edge and connect the vertices of the missing edge to the 6-th vertex. Giving us a desired counterexample.

Remark. Just giving the example and arguing it satisfies the conditions is a complete proof. The above discussion is trying to explain how to get to it and coincidentally shows the above example is smallest possible.

Remark. Note also that this demonstrates a good strategy for approaching problems in which you are not told the answer. You try to prove it and as you go along the proof this might give you information on how your example should look like.

c): The answer is again no, but before giving an example let us again argue how to find it. We need to have two incident edges e and f. One way to ensure they are not in any Euler tour is if their removal disconnects their shared vertex $v = e \cap f$ from their other vertices $u = e \setminus v, w = f \setminus v$, provided the component of v after removal is not just v. So if we take any Eulerian graph H and pick some vertex of it as v and add two new vertices u, w adjacent to v we only need to ensure u and w have an odd degree in the graph obtained by removing H. The easiest way to do it is to simply connect them by an edge, but any graph with an Euler trail from u to w would work. Now for the actual proof we take the simplest example: two triangles intersecting in a single vertex only. The joint vertex is v and edges incident to v in one of the triangles are e and f. This is an Eulerian graph since all degrees are even and it is connected, but if we had an Euler tour which passes through e and f consecutively then it could never reach vertices of the other triangle.

Remark. Note that while arguably we have already proved that our example is a counterexample in the discussion leading to finding it this is no way of writing a proof! It is a good practice in the exam to leave such discussion and mark it as irrelevant for the proof, just in case you make a mistake and we need to assign you a partial grade. Note also that when providing a counterexample you need to prove it is indeed a counterexample if you want all the points.