# Graph Theory

## Solutions 7

**Problem 1:** Let $G$ be a connected graph on $n$ vertices with minimum degree $\delta$. Show that

(a) if $\delta \leq \frac{n-1}{2}$ then $G$ contains a path of length $2\delta$, and

(b) if $\delta \geq \frac{n-1}{2}$ then $G$ contains a Hamiltonian path.

**Solution: a):** The proof mimics the one used in the proof of Dirac's theorem very closely.

- Let us take a longest path $P = v_1 \ldots v_k$ in $G$.
- We may assume for the sake of contradiction that $k \leq 2\delta$.
- We know that $v_1$ and $v_k$ must have all their neighbours belonging to $P$ as otherwise we could find a longer path.
- If there is an $i$ such that $v_1 \sim v_{i+1}$ and $v_k \sim v_i$ then $v_1 v_2 \ldots v_i v_k v_{k-1} \ldots v_{i+1} v_1$ make a cycle $C$ of length $k$.
- Since $n \geq 2\delta + 1$ and $G$ is connected, this cycle must send an edge outside $C$. Removing one of the edges of the cycle incident to this edge we obtain a path on $k+1$ vertices, a contradiction.
- This means that $N(v_1)$ and $N(v_k)^+ := v_j \mid v_{j-1} \sim v_k$ (which is well defined since $N(v_1), N(v_k) \subseteq P$) must be disjoint. Note that $|N(v_1)|, |N(v_k)^+| = |N(v_k)| \geq \delta$ but $N(v_1), N(v_k)^+ \subseteq P \setminus \{v_1\}$. But we can't have two disjoint sets of size $\delta$ living within a set of size at most $k - 1 \leq 2\delta - 1$.

**Remark.** While in the exam you will have access to the lecture notes so we do not ask you to prove theorems you were taught in class, this example shows why knowing the proofs (or at least ideas behind them) can be very useful. This would be a very good exam problem in our opinion.
**Remark.** This method is called rotation-extension or Pósa rotations.

**b):** Let $H$ be the graph obtained from $G$ by adding to it a vertex $v$ connected to every vertex of $G$. Note that $|V(H)| = n+1$ and that $H$ has minimum degree $\delta(H) \geq \delta + 1 \geq \frac{n-1}{2} + 1 =$

$\frac{n+1}{2} = |V(H)|/2$. Therefore, by Dirac's theorem, $H$ contains a Hamiltonian cycle $C$. Removing the vertex $v$ from $H$ we obtain a Hamiltonian path $P = C \setminus \{v\}$ in $G$.

**Problem 2:** Show that the maximum number of edges in a non-Hamiltonian graph on $n \geq 3$ vertices is $\binom{n-1}{2} + 1$.

**Solution:** The graph that is the union of $K_{n-1}$ and an edge attaching it to the $n$'th vertex shows that there is a non-Hamiltonian graph with $\binom{n-1}{2} + 1$ edges. (It's non-Hamiltonian because the degree of the $n$'th vertex is less than 2.)

Now we need to show that any graph $G$ containing at least $\binom{n-1}{2} + 2$ edges (i.e., missing at most $n-3$ edges) is Hamiltonian. We use Ore's condition for this. So let $u, v$ be two non-adjacent vertices, we need to show that $d(u) + d(v) \geq n$. $K_n$ contains $2(n-2) = 2n-4$ edges connecting $u$ and $v$ to the other vertices. But $G$ misses at most $n-4$ of them (the edge $uv$ is already missing), so contains at least $n$ of these edges. This is exactly what we wanted to show.

**Problem 3:**

(a) Show that if $n$ is even, then for every $0 \leq d \leq n-1$, there exists a $d$-regular graph on $n$ vertices.

(b) For odd $n$, show that there exists a $d$-regular graph on $n$ vertices for every **even** $0 \leq d \leq n-1$.

**Solution:** We start with the first item. So assume that $n$ is even. It is sufficient to show that for every $\frac{n}{2} \leq d \leq n-1$, there is a $d$-regular graph on $n$ vertices. Indeed, for $0 \leq d \leq \frac{n}{2} - 1$, we have $n - 1 - d \geq \frac{n}{2}$. Therefore, we will know that there is an $(n-1-d)$-regular graph $\bar{G}$ on $n$ vertices, and then the complement $\bar{G}$ is $d$-regular.

To prove our claim, we show that for $\frac{n}{2} \leq d \leq n-1$, a $d$-regular graph $G$ on $n$-vertices contains a $(d-1)$-regular spanning subgraph. Indeed, by Dirac's theorem, $G$ contains a Hamilton cycle. Taking every second edge of this cycle gives a perfect matching (as $n$ is even). So let $M \subseteq E(G)$ be a perfect matching. Then $G \setminus M$ is $(d-1)$-regular, because every vertex is incident to one edge in $M$. Starting with the complete graph $K_n$ and applying this argument repeatedly, we obtain a $d$-regular $n$-vertex graph for every $\frac{n}{2} - 1 \leq d \leq n-1$.

The solution for the second item is similar. Here we cannot have a perfect matching (because $n$ is odd), so we will remove the edges of Hamilton cycle. This will decrease the degrees by 2. Using Dirac's theorem, this implies that for every even $\frac{n}{2} \leq d \leq n-1$, every $d$-regular graph on $n$ vertices contains a $(d-2)$-regular spanning subgraph. Now start with $K_n$, which is $(n-1)$-regular, and apply this argument repeatedly. Note that $n-1$ is even. This way we get a $d$-regular graph on $n$ vertices for every even $\frac{n-1}{2} \leq d \leq n-1$. To handle $0 \leq d \leq \frac{n-1}{2}$, take complements, noting that $d$ is even if and only if $n-1-d$ is even.

**Remark.** To have a $d$-regular graph on $n$ vertices, it is necessary that $n$ is even or $d$ is even. So Items (a) and (b) cover all possible cases.