# Graph Theory

## Instructor: Oliver Janzer

# Assignment 8

**Please submit your solution to Problem 3 by the end of November 11th for feedback.**

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

**Problem 1:** Let $G$ be a connected graph on more than 2 vertices such that every edge is contained in some perfect matching of $G$. Show that $G$ is 2-edge-connected.

**Problem 2:**

(a) Let $G$ be a graph on $2n$ vertices that has exactly one perfect matching. Show that $G$ has at most $n^2$ edges.

[Hint: .segde gnihctam owt neewteb gnissorc segde owt tsom ta sah G taht evresbO]

(b) Construct such a $G$ containing exactly $n^2$ edges for any $n \in \mathbb{N}$.

**Problem 3:** Let $A$ be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n$ be positive integers. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \in [n]$ if and only if

$$|\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$$

for all $I \subseteq [n]$.

**Problem 4:** Let $G$ be a bipartite graph with sides $X, Y$. Let $A \subseteq X, B \subseteq Y$. Suppose there is a matching $M_A$ which covers all vertices in $A$, and a matching $M_B$ which covers all vertices in $B$. Show that there is a matching which covers all vertices in $A \cup B$. (We say that a matching $M$ covers a set of vertices $U$ if every vertex in $U$ belongs to some edge of $M$.)

**Problem 5:** Suppose $M$ is a matching in a bipartite graph $G = (A \cup B, E)$. We say that a path $P = a_1 b_1 \cdots a_k b_k$ is an *augmenting path* in $G$ if $b_i a_{i+1} \in M$ for all $i \in [k-1]$ and $a_1$ and

$b_k$ are not covered by $M$. The name comes from the fact that the size of $M$ can be increased by flipping the edges along $P$ (in other words, taking the symmetric difference of $M$ and $P$): by deleting the edges $b_i a_{i+1}$ from $M$ and adding the edges $a_i b_i$ instead.

(a) Prove Hall's theorem by showing that if Hall's condition is satisfied and $M$ does not cover $A$, then there is an augmenting path in $G$.

(b) Show that if $M$ is not a maximum matching (i.e. there is a larger matching in $G$) then the graph contains an augmenting path. Is this true for non-bipartite graphs as well?