Graph Theory

Solutions 8

Problem 1: Let G be a connected graph on more than 2 vertices such that every edge is contained in some perfect matching of G. Show that G is 2-edge-connected.

Solution: Suppose for sake of contradiction that G is not 2-edge-connected. Then there exists an edge $uv \in E(G)$ such that removing it leaves 2 connected components U and V with $u \in U$ and $v \in V$. Now let M be a perfect matching containing uv, which exists by assumption. Since there are no edges between U and V besides uv, all vertices of $U \backslash u$ are matched to other vertices of $U \backslash u$ in M, so that $|U \backslash u|$ is even, and similarly $|V \backslash v|$ is even.

Since G has at least 3 vertices, one of U or V must have at least 2 vertices, so without loss of generality lets say it is U. Since U is a connected component there must be a vertex $w \in U$ such that uw is an edge. Now let M' be a perfect matching (in G) containing uw, which again exists by assumption. Note that surely $uv \notin E(M')$. Since there are no edges between U and V besides uv, all vertices of $U \backslash \{u, w\}$ are matched to other vertices of $U \backslash \{u, w\}$ in M', so that $|U \backslash \{u, w\}|$ is even. However, this contradicts the fact that $|U \backslash u|$ was even.

Alternative solution: Consider any edge uv. By assumption, there exists a perfect matching M containing uv. Moreover, since G is connected and has at least 3 vertices, one of u or v must have another neighbour. WLOG say $uw \in E(G)$. Then by assumption, uw is contained in a perfect matching M'. Now, let H be the subgraph of G whose edge set is the symmetric difference of M and M'. Since, $uv \notin M'$, we have $uv \in E(H)$. In general (for arbitrary matchings), each component of H is a path or a cycle, where edges of M, M' alternate.[1] Now, in the special case of starting with two perfect matchings, we claim that there are in fact no paths (except trivial ones consisting of just one vertex). Indeed, suppose the last edge of some component path belongs to $M \backslash M'$. But since M' is perfect, there is some edge at this vertex incident with M'. Moreover, since the last edge belonged to M, this new edge is surely not in M. Hence, it is contained in H, a contradiction. Thus, we have shown that all components of H are isolated vertices or cycles. Since $uv \in E(H)$, this means that uv is contained in a cycle in G, and hence removing it thus not disconnect the graph.

Problem 2:

(a) Let G be a graph on $2n$ vertices that has exactly one perfect matching. Show that G has at most n^2 edges.

[1] In particular, the cycles have even length.

(b) Construct such a G containing exactly n^2 edges for any $n \in \mathbb{N}$.

Solution: (a): Let $u_1v_1, \ldots, u_nv_n$ be the edges of the unique perfect matching M. As indicated by the hint, we claim that for all $i < j$, there are at most 2 edges between $\{u_i, v_i\}$ and $\{u_j, v_j\}$. Indeed, otherwise we would have $u_iu_j \in E(G)$ and $v_iv_j \in E(G)$ or we would have $u_iv_j \in E(G)$ and $v_iu_j \in E(G)$. In either case, we could swap these edges with u_iv_i and u_jv_j in M to obtain a different matching, contradicting the uniqueness of M.

Now, we use this 'local' upper bound to deduce the desired global bound. Every edge in G is either one of the n edges of M, or it is between a pair of matching edges $\{u_i, v_i\}$ and $\{u_j, v_j\}$ for $i < j$. There are $\binom{n}{2}$ such pairs $i < j$, and with the above claim, we infer that in total there are at most $2\binom{n}{2} = n^2 - n$ edges of the latter type. Together with the n edges of M, this proves that G has at most n^2 edges.

(b): We use induction. The main idea is that, if a graph has a vertex v with degree 1, then any perfect matching has to contain the edge incident to this vertex. Now, if we remove v and its neighbour w, we want that the remaining graph has a unique perfect matching. Note that no matter how many neighbours w has, in a perfect matching, it will always be matched with v. This allows us to create a graph with many edges which still has only 1 perfect matching.

Formally, we prove by induction that for all $n \in \mathbb{N}$, there exists a graph G_n with $2n$ and n^2 edges which has a unique perfect matching.

For $n = 1$, we simply define G_1 to be the graph with two vertices that are connected by an edge. Now for $n \geq 2$, assume that G_{n-1} exists. We define G_n by adding two vertices u_n, v_n and adding all edges from u_n to all other vertices (including v_n). Clearly, G_n has $2(n-1)+2 = 2n$ vertices. Moreover, we have added $2(n-1)+1$ edges, so G_n has $(n-1)^2 + (2n-1) = n^2$ edges, as required. Finally, G_n has exactly 1 perfect matching. Indeed, the only edge touching v_n is u_nv_n so it must be part of any perfect matching, and then removing u_n and v_n, we are left with the subgraph G_{n-1}, which has exactly 1 perfect matching by induction.

Problem 3: Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n$ be positive integers. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \in [n]$ if and only if

$$|\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$$

for all $I \subseteq [n]$.

Solution: If there are such subsets then the inequality clearly holds: $\cup_{i\in I} A_i \supseteq \cup_{i \in I} D_i$ by construction for all $I \subseteq [n]$, and $|\cup_{i\in I} D_i| = \sum_{i \in I} |D_i| = \sum_{i\in I} d_i$ because the D_i are disjoint.

To show the other direction, let us define the following bipartite graph. One part will simply be the set A. The other part consists of auxiliary vertices. For every $i \in [n]$, let B_i be a set of size d_i, such that all these sets are disjoint and disjoint from A. Now, let B be the union of these sets B_i. We connect every vertex $v \in B_i$ to all vertices in $A_i \subseteq A$. Our aim is to show that this graph has a matching M covering B. Then taking $D_i = N_M(B_i)$ as the matched image of B_i will work: by construction, $D_i \subseteq A_i$, and since M is a matching, the D_i will be disjoint and will have size $|B_i| = d_i$.

So we just need to check that Hall's condition holds. Take any set $S \subseteq B$. Define $I = \{i : B_i \cap S \neq \emptyset\}$, then clearly $S \subseteq \cup_{i\in I} B_i$. On the other hand, $N(S) = \cup_{i\in I} A_i$ by definition of the graph. But then the condition of the problem gives us what we want:

$$|N(S)| = |\cup_{i\in I} A_i| \geq \sum_{i\in I} d_i = |\cup_{i\in I} B_i| \geq |S|.$$

Problem 4: Let G be a bipartite graph with sides X, Y. Let $A \subseteq X, B \subseteq Y$. Suppose there is a matching M_A which covers all vertices in A, and a matching M_B which covers all vertices in B. Show that there is a matching which covers all vertices in $A \cup B$. (We say that a matching M covers a set of vertices U if every vertex in U belongs to some edge of M.)

Solution: Let M_A be a matching covering A and let M_B be a matching covering B. Let H be the union of M_A and M_B. Then every connected component of H is an even cycle or a path (H may have cycles of length 2, namely pairs of parallel edges). Also, in each such cycle/path, the edges alternate between M_A and M_B. Define a matching M as follows. For each component X of H which is an even cycle or a path with an even number of vertices, take a perfect matching of X and add it to M. Now consider a connected component X which is a path with an odd number of vertices, hence an even number of edges. Let $x_1, \ldots, x_k$ be the vertices of this path, k odd. Suppose without loss of generality that $x_1x_2 \in M_A$. As the edges alternate between M_A and M_B, and k is odd, we have $x_{k-1}x_k \in M_B$. This means that $x_k \notin A$, because $x_{k-1}x_k$ is the only edge of $H = M_A \cup M_B$ containing x_k, so x_k is not covered by M_A. Now add to M the matching $x_1x_2, x_3x_4, \ldots, x_{k-2}x_{k-1}$. This matching covers all vertices of $X \setminus \{x_k\}$, hence all vertices of $X \cap (A \cup B)$. Doing this for every connected component X gives the required matching M.

Problem 5: Suppose M is a matching in a bipartite graph $G = (A \cup B, E)$. We say that a path $P = a_1b_1 \cdots a_kb_k$ is an *augmenting path* in G if $b_ia_{i+1} \in M$ for all $i \in [k-1]$ and a_1 and b_k are not covered by M. The name comes from the fact that the size of M can be increased by flipping the edges along P (in other words, taking the symmetric difference of M and P): by deleting the edges b_ia_{i+1} from M and adding the edges a_ib_i instead.

(a) Prove Hall's theorem by showing that if Hall's condition is satisfied and M does not cover A, then there is an augmenting path in G.

(b) Show that if M is not a maximum matching (i.e. there is a larger matching in G) then the graph contains an augmenting path. Is this true for non-bipartite graphs as well?

Solution: (a): Given a matching M in G, let us call a path in G *alternating* if it alternates between edges from M and outside M. We want to show that if G satisfies Hall's condition and M does not cover A then there is an alternating path connecting two unmatched vertices $a \in A$ and $b \in B$.

We define the following orientation of G: edges in M are oriented from B towards A, whereas edges outside M point from A to B. Fix any unmatched vertex $a \in A$. Let S be the set of vertices that can be reached from a with a directed path. Then S is exactly the set of vertices s such that an alternating path between a and s exists. Let $A' = A \cap S$ and $B' = B \cap S$. We need to show that B' contains an unmatched vertex.

Consider A'. Clearly no non-matching edge uv can exist between $A' \ni u$ and $B - B' \ni v$, because then v could be reached from a via u, so v would be in $S \cap B = B'$. We also claim that there are no matching edges between A' and $B - B'$. Indeed, consider any vertex $a' \in A'$. If $a' = a$, then a' is unmatched and so there clearly is no matching edge to $B - B'$. If $a' \neq a$, then the last edge of the directed path from a to a' was a matching edge, coming from $b \in B$. However, the subpath from a to b also certifies that $b \in S$. Hence, a' is matched to a vertex in B', and so surely there cannot be another matching edge to $B - B'$.

Altogether, we have shown that there is no edge at all between A' and $B - B'$. Hence, $N(A') \subseteq B'$. Now, by Hall's condition, we know that $|B'| \geq |N(A')| \geq |A'|$.

Finally, observe that the number of matched vertices in A' is at least as large as the number of matched vertices in B'. Indeed, for any matched vertex b in B', the directed path from a to b ends with a non-matching edge, and hence we can use the matching edge at b to extend the path and see that b's match also belongs to S and hence to A'. Since we use matching edges for these extensions, all the vertices in A' obtained in this way are distinct, proving the claim.

Now, we are done, since A' contains the unmatched vertex a, B' must also contain an unmatched vertex b. This shows the existence of an augmenting path from a to b.

Now we use this to give a new proof of Hall's theorem. That Hall's condition is necessary for the existence of a matching covering A is clear. Now, assume that Hall's condition holds. Starting with the empty matching, we can iteratively find augmenting paths and extend the current matching along it (as described in the problem statement), until we reach a matching covering A.

Remark: Observe that a single edge connecting two unmatched vertices is also an augmenting path (check the definition). This seems to have confused some students.

(b): We give a proof that also works for non-bipartite G.

Let N be a larger matching than M and consider the subgraph H defined by symmetric difference of the edges of M and N, i.e. $V(H) = V(G)$ and

$$E(H) = E(M) \triangle E(N) = (E(M) \cup E(N)) \setminus (E(M) \cap E(N)).$$

Since M and N are graphs with maximum degree at most 1, H has maximum degree at most 2. It follows that H must be a disjoint union of isolated vertices, paths, and cycles.

Note that $E(H)$ is the disjoint unions of $E(M) \setminus E(N)$ and $E(N) \setminus E(M)$. Now color the edges of $E(M) \setminus E(N)$ red and the edges of $E(N) \setminus E(M)$ blue. Since M and N are matchings, each path and cycle of H must alternate in colors, as no vertex can be incident with two edges of the same colour. In particular, each cycle is of even length and has the same number of blue as red edges. However, since N was a larger matching than M, H must have more blue than red edges. Therefore, H must have at least one path with more blue than red edges. This means that the path has odd length and that both the first and last edge belong to N but not M. Let v be an endpoint of this path. Then, v is not covered by M. Indeed, otherwise the edge in M covering it would also be in N by definition of H, but then v is incident to at least two edges in N, contradicting that N is a matching. Since the path alternates between edges that are in and not in M, this is an augmenting path for M.

Remark: Instead of considering the symmetric difference, one can also work with the union. Then, in addition to the above, some components consist of single edges that belong to both matchings.