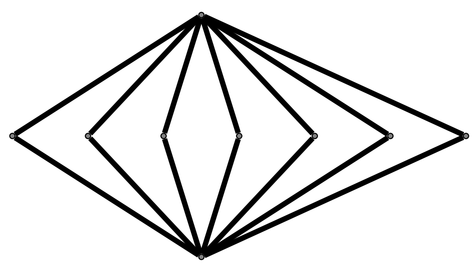
Graph Theory

Solutions 9

Problem 1: Determine all positive integers r and s, with $r \leq s$, for which $K_{r,s}$ is planar.

Solution: By Corollary 6.13. from the lecture notes, $K_{3,3}$ is not planar, so every graph which contains $K_{3,3}$ as a subgraph is not planar. Hence $K_{r,s}$ is not planar whenever both r and s are larger or equal to 3. Suppose now that $r \leq 2$. We will show that $K_{r,s}$ is planar in this case. By doing so we will have covered all possible combinations for r and s.

If we show that $K_{2,s}$ is planar for every s, then we are done, as $K_{1,s}$ is a subgraph of $K_{2,s}$, and a subgraph of a planar graph is also planar. Showing that $K_{2,s}$ is planar is easy, as we only have to show how to draw it without crossings and it can be done as in the following picture:



Problem 2:

(a) Show that every planar graph has a vertex of degree at most 5. Is there a planar graph with minimum degree 5?

(b) Show that any planar *bipartite* graph has a vertex of degree at most 3. Is there a planar bipartite graph with minimum degree 3?

Solution: (a): Let G be any planar graph with $|V(G)| = n > 5$ vertices (with less it is trivial). By Theorem 6.11 from the lecture notes, we know that G has at most $3n-6$ edges. This means that the average degree of G is at most $2\frac{3n-6}{|V(G)|} < 6$, because the average degree is equal to $2\frac{|E(G)|}{n}$ for every graph G. Therefore, there must exist a vertex of degree at most 5, because otherwise all degrees would have degree more than 5, hence the average would be at least 6.
If you try to come up with an example for the second question, very soon you realise that you will need a lot of vertices to construct it, if it exists. One could wrongly conclude that it is maybe better to try to prove that such a graph does not exist. But if one knows the constructions of platonic solids, or at least knows that they exist, then a quick look at the lecture notes

reveals that the icosahedron-graph is such a graph. Furthermore, it can be shown that there are no examples on less than 12 vertices. Try to show it!

(b): From the same theorem as in part (a) we conclude that the average degree of a planar bipartite graph on n vertices is at most $2\frac{2n-4}{n} < 4$, so there exists a vertex of degree at most 3, as before.
An example for the second question is the cube-graph. Find a bipartition of it!

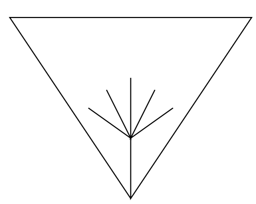
Problem 3: Show that a connected plane graph G is bipartite iff all its faces have even length.

Solution: Let G be a planar graph. We already know that G is bipartite iff all its cycles have even length (first exercise sheet), so it is enough to prove that the statement "every face has even length" is equivalent to these. (In the solution to the first exercise, we also showed that these statements are equivalent to the statement "every closed walk has even length".) We will first show that bipartite implies that all faces have even length, and then that if all faces have even length then all cycles have even length as well.

To see the first statement, let us take any face and a vertex v on it. Now start walking from v along the boundary of the face. Since the graph is bipartite, we alternately encounter vertices from the two classes, and when we reach v again, we are back at its class. So we made an even number of steps. Since the boundary of the face consists of a disjoint union of closed walks (in fact there is only one since the graph is connected), each of which has even length by the above argument, we conclude that the length of the face is even.

For the second statement, assume that all faces have even length, and consider a cycle in the drawing, of length l. The region bounded by the cycle is the union of faces, let the length of those faces be $l_1, \ldots, l_k$. Then the sum $\sum_{i=1}^{k} l_i$ counts each edge on the cycle once, and each edge inside the region bounded by the cycle twice (once for both adjacent faces, which may be the same face). So $\sum_{i=1}^{k} l_i \equiv l \pmod 2$, and since all the l_i are even, l is also even.

Remark. Notice that a face in a plane graph does not have to be a cycle. One example is given in the following picture, where the face inside of the triangle is not bounded by a cycle, but with a walk which repeats edges.



Informal discussion The walk from the first part of the proof for the particular face in the above figure can be visualised as follows - imagine the part of the graph inside of the triangle is your arm (no offence intended). Start from the bottom and go up and suppose you want to draw the contours of your hand on a paper (you go up and down for each of your fingers, thus traversing them twice in the walk). When you are back at the bottom of your arm, traverse the triangle in any direction.

Remark. If the graph is 2-connected then one can show that every face is bounded by a cycle (where the unbounded face is bounded away from the interior of the graph also by a cycle).

Problem 4: Let G be a graph on $n \geq 3$ vertices and $3n - 6 + k$ edges for some $k > 0$. Show that any drawing of G in the plane contains at least k crossing pairs of edges.

Solution: We show this claim by induction. For the base of the induction consider $k = 1$. Since any planar graph has at most $3n - 6$ edges, then a graph with $3n - 6 + 1$ edges is not planar, which means that any drawing of it has at least 1 crossing.
Induction hypothesis: Assume that the claim holds for $k = \ell$, where $\ell \geq 1$.
Induction step: We prove that the claim holds for $k = \ell + 1$. Let G be a graph with $3n - 6 + \ell + 1$ edges and consider an arbitrary drawing of G. Since $3n - 6 + \ell + 1 > 3n - 6$ then the drawing of G contains at least one crossing pair of edges. Remove one of those two crossing edges from G to obtain the graph G'. Since G' has $3n - 6 + \ell$ edges and n vertices, by the induction hypothesis we conclude that the drawing restricted to G' has at least ℓ crossing pairs of edges. Adding back the drawing of the removed edge produces at least one more crossing, so we have at least $\ell + 1$ crossing pairs in an arbitrary drawing of G, so we are done.

Problem 5: Let G be a plane graph with triangular faces and suppose the vertices are colored arbitrarily with three colors. Prove that there is an even number of faces that get all three colors.

Solution: Take an arbitrary (not necessarily proper) 3-coloring of G, and let a_i be the number of 2-colored sides of the i'th face. Note that if the vertices of this i'th triangle get 3 different colors, then $a_i = 3$, if they only get 2 different colors, then $a_i = 2$, and if all the vertices have the same color, then $a_i = 0$.

The sum $\sum a_i$ counts each 2-colored edge twice (once for each adjacent face), and since the a_i's can only take the values 0, 2 or 3, this means we have an even number of 3's among them. That means that there is an even number of 3-colored triangles.

Problem 6: Let S be a set of $n \geq 3$ points in the plane such that any two of them have distance at least 1. Show that there are at most $3n - 6$ pairs of distance *exactly* 1.

Solution: Draw a straight line segment between any pair of points in S that have distance exactly 1. It is enough to show that there are no crossing segments. Indeed, if that is the case, then the drawing corresponds to a plane embedding of a graph, where two points are connected by an edge if they have distance 1. Since a plane graph on $n \geq 3$ vertices has at most $3n - 6$ edges, we get that there are at most $3n - 6$ such pairs of points.

Suppose for contradiction that two of the above segments AB and CD intersect at some point X. Remember, both segments have length $l(AB) = l(CD) = 1$. By the triangle inequality, $l(AX) + l(XC) \geq l(AC)$ and $l(XB) + l(DX) \geq l(BD)$, so

$$l(AC) + l(BD) \leq l(AX) + l(XB) + l(XC) + l(DX) = l(AB) + l(CD) = 2.$$

But by the condition, $l(AC)$ and $l(BD)$ both are at least 1, so we in fact have equality everywhere. Similarly we get that $l(AD)$ and $l(BC)$ are 1, so we have four points in the plane, any two of unit distance, which is impossible. Why? So indeed there is no crossing.