Graph Theory

Solutions 10

Problem 1: Are the following statements true?

(a) If G and H are graphs on the same vertex set, then $\text{dg}(G \cup H) \leq \text{dg}(G) + \text{dg}(H)$.

(b) If G and H are graphs on the same vertex set, then $\chi(G \cup H) \leq \chi(G) + \chi(H)$.

(c) Every graph G has a $\chi(G)$-coloring where $\alpha(G)$ vertices get the same color.

Solution:

a) The answer is no. We may take G and H to be edge disjoint paths of length 3, so $G \cup H = K_4$. We have $dg(G \cup H) = 3$ but $dg(G) = dg(H) = 1$ and $3 > 1 + 1$. If you had a proof of this being true please go through it carefully and figure out what goes wrong (it is a very very good exercise). A way to get to the example is to try to prove it and find where you get stuck.

b) It is a very tempting claim and we usually see a large number of "proofs". However, **the answer is no** it is not true. We take graphs with 6 vertices, for G we take a disjoint union of triangles and for H the $K_{3,3}$ consisting of all edges joining the triangles making G. Then $G \cup H = K_6$ and has $\chi(G \cup H) = 6$ but $\chi(G) = 3, \chi(H) = 2$ and $6 > 2 + 3$. If you had a proof of this being true please go through it carefully and figure out what goes wrong (it is a very very good exercise). What is true is $\chi(G \cup H) \leq \chi(G) \cdot \chi(H)$, also if G and H are vertex disjoint then the claim holds. Prove both of these claims as exercise.

c) The answer is yet again no. Let us take G to be two vertex disjoint stars (say with 2 leaves each) with centres joined by an edge. Then $\chi(G) = 2$, yet in any 2-colouring centres need to get distinct colours so also all leaves of one star need to be coloured using the same colour, distinct between stars. This means each colour class has size exactly 3 in any 2-colouring. Yet $\alpha(G) = 4$ (take all leaves of both stars). A very good place to start looking for the counterexample above is the inequality $\alpha(G)\chi(G) \geq n$ which might be very far from equality. What is true is that there is a $\Delta + 1$-colouring in which size of every colour class differs by at most one (this is called the Hajnal-Szemerédi Theorem) so they are all of size roughly $n/\chi(G)$.

Problem 2: G has the property that any two odd cycles in it intersect (they have at least one vertex in common). Prove that $\chi(G) \leq 5$.

Solution:

- Let us take an odd cycle C of minimum size, so C is induced, in other words there are no edges between vertices of C except the ones which are part of C (we have seen this a few times now, try to remember why, also note that this crucially relies on the cycle being minimal!).

- Let us now delete all vertices of C to obtain G'.

- By assumption any two odd cycles in G intersect G' contains no odd cycles (any such cycle would not intersect C). Hence, G' is by Exercise 7 from homework 1 bipartite, i.e. $\chi(G') \leq 2$.

- On the other hand $\chi(C) \leq 3$. So we may use two colours to colour G' and 3 new colours to colour G'. Note that we are crucially using the fact that C has no chords!!! The inequality we are using is the correct form of Problem 1b. If we split the vertices of G into subsets S and Q then $\chi(G) \leq \chi(G[S]) + \chi(G[Q])$.

Problem 3: For a vertex v in a connected graph G, let G_r be the subgraph of G induced by the vertices at distance r from v. Show that $\chi(G) \leq \max_{0 \leq r \leq n} \chi(G_r) + \chi(G_{r+1})$.

Solution: We need to color the vertices of G using $m = \max \chi(G_r) + \chi(G_{r+1})$ colors. The main idea that we need is the fact that there is no edge between G_r and G_{r-2} for any r, so we can color them independently of each other.

We color the vertices inductively. First we color $G_0 = v$ with some color. Then we have $m - \chi(G_0)$ colors left to use for G_1. Since $\chi(G_1) \leq m - \chi(G_0)$, we can properly color G_1 using $\chi(G_1)$ colors that all differ from the colors of G_0. And so on: in the r'th step we color G_r using $\chi(G_r)$ colors that are all different from the ones we used to color G_{r-1} (we can do this because $\chi(G_r) \leq m - \chi(G_{r-1})$. And this will be a proper coloring, because all edges of G either go within some G_r, or between some G_{r-1} and G_r. (And, since G is connected, we color all the vertices eventually.)

Problem 4: Let l be the length of the longest path in a graph G. Prove $\chi(G) \leq l+1$ using the fact that if a graph is not d-degenerate then it contains a subgraph of minimum degree at least $d+1$.

Solution: If G is l-degenerate, then $\chi(G) \leq l+1$. So suppose $\chi(G) > l+1$. Then G is not l-degenerate, hence contains some subgraph of minimum degree at least $l+1$. But as we saw in the first week of the course, a graph of minimum degree $l+1$ contains a path of length $l+1$. Contradiction.

Problem 5: Suppose the complement of G is bipartite. Show that $\chi(G) = \omega(G)$.

Solution: The chromatic number of G is the minimum number of independent sets covering the vertices, so it is the minimum number of complete graphs in $\overline{G}$ covering all the vertices. As $\overline{G}$ is bipartite, its cliques are singletons and edges — to have as few cliques as possible, we want to use as many edges as we can. This means that $\chi(G) = n - \nu(\overline{G})$, where n is the number of vertices.

$\omega(G) = \alpha(\overline{G})$, so we want to have a large independent set. But an independent set I spans no edges, so $V - I$ covers all the edges. This means that large independent sets correspond to small edge-covering sets, i.e. $\alpha(\overline{G}) = n - \tau(\overline{G})$. But from Kőnig's theorem, we know $\nu(\overline{G}) = \tau(\overline{G})$ for bipartite $\overline{G}$, so we are done.