Graph Theory

Solutions 11

Problem 1: For a given natural number n, let G_n be the following graph with $\binom{n}{2}$ vertices and $\binom{n}{3}$ edges: the vertices are the pairs (x, y) of integers with $1 \leq x < y \leq n$, and for each triple (x, y, z) with $1 \leq x < y < z \leq n$, there is an edge joining vertex (x, y) to vertex (y, z). Show that for any natural number k, the graph G_n is triangle-free and has chromatic number $\chi(G_n) > k$ provided $n > 2^k$.

Solution: We have to show two things: that G_n is triangle-free and that $\chi(G_n) > k$ for $n > 2^k$.

We first show that G_n is triangle-free. This part does not need induction, but can be proven directly. Let us assume for the sake of contradiction that there is a triangle $(x, y), (z, w), (p, q) \in V(G_n)$ in G_n. Assume without loss of generality that $x \leq z \leq p$. Note that in order for (x, y) and (z, w) to be connected by an edge we need to have $y = z$ and similarly for the other pairs to be edges we must have $w = p$ and $y = p$. So we obtain $w = p = y = z$ which is a contradiction since $z < w$.

Now, we turn to the chromatic number. We will show the following contrapositive of the claim in the problem, using induction on k : If $\chi(G_n) \leq k$ then $n \leq 2^k$.

To apply induction, we will need the following crucial 'self-similarity' property of the sequence (G_n): for any subset $X \subset [n]$, let $G_n \mid_X$ denote the subgraph of G_n induced by the set of all (x, y) with $x < y$ and $x, y \in X$. Then $G_n \mid_X \cong G_{|X|}$. Indeed, if we consider the elements in X in their natural order and relabel the ith element with i, then we obtain $G_{|X|}$.

Now, we start the induction proof. For the base case $k = 1$, note that for any graph G, $\chi(G) = 1$ is only possible if G has no edges. As G_n has edges for $n \geq 3$ (say $\{(1, 2), (2, 3)\}$) this gives that if $\chi(G_n) \leq 1$ then $n \leq 2$. We now assume that the induction hypothesis holds for $k - 1$; we want to show that it holds for k. Assume $\chi(G_n) \leq k$, and let c be a proper k-colouring of G_n. Let

$$A = \{x \in [n] \mid \text{ there is no } y > x \text{ such that } c((x, y)) = k\}$$

$$B = \{x \in [n] \mid \text{ there is no } y < x \text{ such that } c((y, x)) = k\}.$$

First, we show that every vertex belongs to A or B.

$$[n] = A \cup B$$

Assume for the sake of contradiction that there is an $x \in [n]$ such that $x \notin A \cup B$. $x \notin A$ implies that there is a $y > x$, such that $c((x, y)) = k$, while $x \notin B$ implies there is a $z < x$, such that $c((z,x)) = k$. But since $z < x < y$, we have that $\{(z,x),(x,y)\}$ is an edge in G_n, so this contradicts the fact that c was a proper colouring.

Recall that we want to show that $n \le 2^k$. Clearly, if we could show that both A and B have size at most 2^{k-1}, then the assertion would follow from the above claim. In order to achieve this, we want to use induction on the two induced subgraphs $G_n\mid_A$, $G_n\mid_B$ (notation as above).

$$|A|, |B| \le 2^{k-1}$$

Note that the colouring c when restricted to $G_n\mid_A$ does not use colour k, since if $c((x,y)) = k$ then $x \notin A$. Hence, c gives a proper $k-1$-colouring of $G_n\mid_A$, and as $G_n\mid_A$ is isomorphic to $G_{|A|}$ as observed above, we have by induction that $|A| \le 2^{k-1}$. An analogous argument implies $|B| \le 2^{k-1}$, where we use that $c((x,y)) = k$ implies $y \notin B$.

Combining these two claims we obtain:

$$n \le |A \cup B| \le |A| + |B| \le 2^{k-1} + 2^{k-1} = 2^k,$$

so the induction hypothesis holds for k.

Alternative solution:

Check that G_n is triangle-free as before. Now, assume that $\chi(G_n) \le k$ and let c be a proper k-coloring. For $x \in [n]$, define $S_x = \{c(u,x) \colon 1 \le u < x\} \subset [k]$. We claim that the sets $S_1, \dots, S_n$ are pairwise distinct. Suppose for a contradiction that $S_x = S_y$ for $x < y$. Clearly, $c(x,y) \in S_y$ by definition of S_y. By the assumption, we also have $c(x,y) \in S_x$, meaning that there exists $1 \le u < x$ such that $c(u,x) = c(x,y)$, which is a contradiction since (u,x) is connected to (x,y).

Since the total number of subsets of $[k]$ is 2^k, it follows that $n \le 2^k$.

Problem 2: A proper edge-coloring of a graph G is an assignment of colors to the edges of G such that no two edges with a common endpoint have the same color. The edge-chromatic number of G is the minimum number of colors in a proper edge-coloring of G. Find the edge-chromatic number of K_n.

Solution: We denote the edge-chromatic number of K_n by $\chi'(K_n)$. Since every edge incident to a given vertex must have a different color in a proper edge-coloring, we have $\chi'(K_n) \ge \Delta(K_n) = n-1$. Next, we show that $\chi'(K_n) \le n$. We do this by finding an explicit proper edge-coloring of K_n using n colors. There are many ways to do this. One easy description uses the group $\mathbb{Z}_n$. Identify the vertices of K_n with $\mathbb{Z}_n$, and define an edge-colouring c by $c(xy) := x + y$. Clearly, this coloring uses n colors. Moreover, we have $c(xy) \ne c(xz)$ whenever $y \ne z$, so the coloring is proper.

Hence, we have shown that $n-1 \leq \chi'(K_n) \leq n$. We now claim that for odd n, the upper bound is correct, whereas for even n, the lower bound is correct.

Suppose first that n is odd. Then any matching can only contain $\lfloor n/2 \rfloor = \frac{n-1}{2}$ edges. Observe that each color class in a proper edge-coloring forms a matching, and since the total number of edges is $n\left(\frac{n-1}{2}\right)$, we need at least n colours to edge-color K_n if n is odd. This, together with the explicit coloring defined above, shows that $\chi'(K_n) = n$ if n is odd.

Finally, assume that n' is even. Then $n = n' - 1$ is odd. Consider the colouring of the complete graph on $\mathbb{Z}_n$ as given above. This uses $n = n' - 1$ colours. Note that for vertex x, there is no edge at x with colour $2x$, since $x + y = 2x$ implies $y = x$. We now add a new vertex a to the graph (and all edges) and want to extend the n-edge-coloring by defining colors for the edges at a. Since we already observed that vertex x has no edge with color $2x$, a good idea is to color edge xa with color $2x$. Moreover, since $2x \neq 2y$ whenever $x \neq y$ (all calculations are in $\mathbb{Z}_n$, the conclusion holds because 2 and n are coprime since n is odd), we do not use a color twice this way. Hence, we have obtained an $(n'-1)$-edge-coloring of $K_{n'}$. Thus, $\chi'(K_n) = n-1$ if n is even.

(It holds more generally that if n is odd, any n-edge-coloring of K_n can be extended to a an n-edge-coloring of K_{n+1}, since each vertex 'misses' a unique color, and these colors are distinct for distinct vertices (to show this one needs that n is odd). Hence, we can simply use these missing colors to color the edges to the new vertex.)