Graph Theory

Solutions 12

Problem 1: The lower bound for $R(p,p)$ that you learn in the lectures is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_p.

Give an explicit coloring on $K_{(p-1)^2}$ that proves $R(p,p) > (p-1)^2$.

Solution: Take $(p-1)^2$ vertices and split them into $p-1$ equal groups. Now we color each edge induced by some of the groups red, and color edges between different groups blue. We claim that this edge-coloring of the graph contains no monochromatic clique of size p. Indeed, among any p vertices, we will have two in the same group (by the pigeonhole principle), so any p vertices induce a red edge. On the other hand, two of the vertices must come from different groups, so a blue edge is also induced. Hence this construction shows $R(p,p) > (p-1)^2$.

Problem 2: Show that every red/blue-colouring of the edges of K_{6n} contains n vertex-disjoint triangles with all $3n$ edges of the same colour.

Solution: First we find $2n-1$ monochromatic triangles. To do this, take 6 vertices, and use $R(3,3) = 6$ to find a monochromatic triangle in them. Then remove the 3 vertices of the monochromatic triangle and take 6 new vertices. The last step of this process is when we have 6 vertices left (so after this step we will only have 3 left). This way, we get $2n-1$ monochromatic triangles (and 3 extra vertices). By the pigeonhole principle, n of the triangles have the same colour.

Problem 3:

(a) Let $n \geq 1$ be an integer. Show that any sequence of $N \geq R(n,n)$ distinct numbers, $a_1, \ldots, a_N$ contains a monotone (increasing or decreasing) subsequence of length n.

(b) Let $k, l \geq 1$ be integers. Show that any sequence of $kl+1$ distinct numbers $a_1, \ldots, a_{kl+1}$ contains a monotone increasing subsequence of length $k+1$ or a monotone decreasing subsequence of length $l+1$.

Solution: (a): Colour the edges of he complete graph G on vertex set $[N]$ as follows: for $i < j$, colour the edge ij red, if $a_i < a_j$, otherwise colour ij blue. As $N \geq R(n,n)$, G contains a monochromatic clique of size n. Let $i_1 < ... < i_n$ be the vertices of such a clique. If the clique is red, then $a_{i_1} < ... < a_{i_n}$, otherwise $a_{i_1} > ... > a_{i_n}$.

(b): Let s_i be the length of the longest decreasing subsequence starting at a_i. Notice that if $s_i = s_j$ for some $i < j$ then $a_i < a_j$. Indeed, otherwise we could add a_i to the decreasing sequence of length s_j starting at a_j and get a sequence of length $s_i + 1$ starting at a_i, contradicting our definition.

Now if there is no decreasing subsequence of length $l + 1$, then all the s_i are in $\{1, \ldots, l\}$. There are $kl+1$ numbers, so some value $x \in \{1, 2, \ldots, l\}$ is taken by at least $k+1$ of the s_i. But by the above observation, these s_i will correspond to an increasing sequence of length $k + 1$, which is what we wanted to show.