Graph Theory

Solutions 13

Problem 1: Prove that if there is a real number p, $0 \leq p \leq 1$, such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this, show that the following holds, for some constant c.

$$R(4,t) \geq c \cdot \frac{t^{3/2}}{(\log t)^{3/2}}.$$

Solution: Let n be a positive integer and suppose that there exists p as described above. Colour every edge of the complete graph on n vertices, K_n, randomly either red, with probability p, or blue, with probability $1-p$.

For any $K \subseteq V(K_n)$ of size k let X_K be indicator random variable for the event that all the edges in the subgraph of K_n induced by K are coloured red. Analogously, for any $T \subseteq V(K_n)$ of size t let Y_T be indicator random variable for the event that all the edges in the subgraph of K_n induced by T are coloured blue. Notice that $\mathbb{E}[X_K] = p^{\binom{k}{2}}$ for any K as above and $\mathbb{E}[Y_T] = (1-p)^{\binom{t}{2}}$ for any T as above. Hence, by the linearity of expectation,

$$\mathbb{E}\left[\left(\sum_{|K|=k} X_K\right) + \left(\sum_{|T|=t} Y_T\right)\right] = \sum_{|K|=k} \mathbb{E}[X_K] + \sum_{|T|=t} \mathbb{E}[Y_T] = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1,$$

by hypothesis. Therefore, with positive probability $\sum_{|K|=k} X_K + \sum_{|T|=t} Y_T = 0$, which means that there is a way of coloring K_n with the colors red and blue in such a way that there is no red clique of size k nor blue clique of size t. As such, we must have $R(k,t) > n$ by definition of $R(k,t)$.

To show the second part, it suffices to check that if $n \leq c\left(t^{3/2}/(\log t)^{3/2}\right)$ for some absolute constant $c > 0$ then for sufficiently large t there always exists a p satisfying the inequality in the problem with $k = 4$. Take $p = \frac{\log t}{t}$. Clearly $0 \leq p \leq 1$. Also, considering n and p as indicated

$$\begin{aligned}\binom{n}{4} p^{\binom{4}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} &\leq n^4 p^6 + \left(\frac{en}{t}\right)^t e^{-p\binom{t}{2}} \leq c^4 + \frac{(ec)^t t^{t/2}}{(\log t)^{3t/2}} e^{-(t-1)(\log t)/2} \\ &= c^4 + \frac{(ec)^t e^{(\log t)/2}}{(\log t)^{3t/2}} = c^4 + \frac{(ec)^t \sqrt{t}}{(\log t)^{3t/2}} \to c^4,\end{aligned}$$

where we use the assumption that t tends to infinity. Thus, if $c < e^{-1}$ then for t sufficiently large the desired inequality is satisfied.

Problem 2: Prove that for every fixed positive integer r, there is an n such that any coloring of all the subsets of $[n]$ using r colors contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same color.

Solution 1. First, we show that for every triple of positive integers r, l, m there exists $M = M(r, l, m)$ such that the following holds. Let f be an r-coloring of the subsets of $[M]$, then there exists $A \subset M$ such that $|A| = m$ and for $i = 1, \ldots, l$, the subsets of A of size i have the same color.

We prove the existence of M by induction on l. If $l = 1$, then $M = (m-1)r + 1$ trivially suffices. Now suppose that $l \geq 2$, and let $m' = M(r, l-1, m)$. By Ramsey's theorem, there exists M such that for any r coloring of the l-element subsets of $[M]$, there exists $X' \subset [M]$ of size m' such that every l-element subset of X' has the same color. But then X' contains a subset X of size m such that for $i = 1, \ldots, l-1$, the subsets of X of size i have the same color. Hence, this M suffices and we can set $M(r, l, m) = M$.

Now let N be a positive integer such that any r-coloring of $[N]$ contains a monochromatic solution of $a + b = c$ (such an N exists by Schur's theorem).

We show that $n = M(r, N, N)$ suffices. Indeed, for any r-coloring f of the subsets of $[n]$ there exists $A \subset [n]$ of size N such that for $i = 1, \ldots, N$, the i-element subsets of A have the same color. Consider the r-coloring g of $[N]$ where $g(i)$ is the color of i-element subsets of A. Then there exists $a, b, c \in [N]$ such that $g(a) = g(b) = g(c)$ and $a + b = c$. Let X be any a element subset of A, let Y be any b element subset of A disjoint from X (as $a + b \leq |A|$, it is possible to find such Y). Then $X, Y, X \cup Y$ have the same color, finishing the proof.

Solution 2. We show that $n = R_r(3)$ suffices, where $R_r(3)$ is the smallest n such that any r-coloring of the complete graph on n vertices contains a monochromatic triangle.

Let f be an r-coloring of $2^{[n]}$, and let K_n be the complete graph on vertex set $[n]$. Define the r coloring g on K_n such that for $1 \leq x < y \leq n$, $g(x, y) = f(\{x, x+1, \ldots, y-1\})$. Then K_n contains a monochromatic triangle, say with vertices $x < y < z$. Let $X = \{x, \ldots, y-1\}$, $Y = \{y, \ldots, z-1\}$ and $Z = \{x, \ldots, z-1\}$. Then, we have $f(X) = g(x, y), f(Y) = g(y, z)$ and $f(Z) = g(x, z)$. Since the triangle x, y, z is monochromatic under the coloring g, we have $f(X) = f(Y) = f(Z)$. Therefore, the sets X, Y satisfy the requirements, since they are disjoint and $X \cup Y = Z$.

Problem 3: Prove the following strengthening of Schur's theorem: for every $k \geq 2$ there is an N such that any k-coloring of $[N]$ contains three *distinct* integers a, b, c of the same color satisfying $a + b = c$.

Solution 1. Let N the smallest positive integer such that any $2k$-colouring of $[N]$ contains three (not necessarily distinct) numbers a, b, c of the same colour such that $a + b = c$ (N exists by Schur's theorem). We show that N suffices.

Indeed, given a colouring f of $[N]$ with k colours, define the colouring g of $[N]$ with at most $2k$ colours as follows: x can be uniquely written as $2^r q$, where r and q are integers and q is odd. If r is even, let $g(x) = (f(x), 0)$, otherwise let $g(x) = (f(x), 1)$. By the definition of N, there exist $a, b, c \in [N]$ such that $a + b = c$ and $g(a) = g(b) = g(c)$. But then $f(a) = f(b) = f(c)$ as well. It remains to show that a, b, c are distinct, that is, $a \neq b$. Suppose that $a = b = 2^r q$, where q is odd. Then $c = 2^{r+1} q$, and as r and $r + 1$ have different parities, we have $g(a) \neq g(c)$, contradiction.

Solution 2. (hint) Proceed as in the proof of Schur's theorem, but apply Ramsey's theorem for K_4 (instead of K_3). Namely, we are given a k-colouring f of $[N]$, and we colour the edges of K_N with k colours, where the colour of edge (x, y) (with $x < y$) is $f(y - x)$. Now we apply Ramsey's theorem to find a monochromatic K_4 with vertices $x < y < z < w$. (So we need to take $N = R_k(4)$). This can be used to get distinct a, b, c with $a + b = c$ and $f(a) = f(b) = f(c)$ (complete the details!).

Problem 4: Prove that for every $k \geq 2$ there exists an integer N such that every coloring of $[N]$ with k colors contains three distinct numbers a, b, c satisfying $ab = c$ that have the same color.

Solution: Assuming Problem 4, there exists $N_0 = N_0(k)$ such that in any k-colouring of $[N_0]$ one can find a monochromatic triple (a, b, c) of distinct integers such that $a + b = c$. We show that $N = 2^{N_0}$ suffices. Indeed, if f is a colouring of $[N]$, we can define the colouring $g(x) = f(2^x)$ of $[N_0]$. Then, if (a, b, c) is a monochromatic triple in g satisfying $a + b = c$, then $A = 2^a, B = 2^b, C = 2^c$ is a monochromatic triple in f satisfying $AB = C$.

(One can get a different solution by using Problem 2. and considering square-free numbers. Try to work out the details!)