# Graph Theory

## Instructor: Oliver Janzer

## Assignment 14

**Please submit your solution to Problem 2 by the end of January 4th for feedback.**

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

**Problem 1:** Let $H$ be an arbitrary fixed graph and prove that the sequence $\frac{\mathrm{ex}(n,H)}{\binom{n}{2}}$ is (not necessarily strictly) monotone decreasing in $n$.

**Problem 2:** Imitate the proof of Turán's theorem to show that among all the $n$-vertex $K_{r+1}$-free graphs, the Turán graph $T_{n,r}$ contains the maximum number of triangles (for any $r, n \geq 1$).

**Problem 3:** Let $a_1, \dots, a_n \in \mathbb{R}^d$ be vectors such that $|a_i| \geq 1$ for each $i \in [n]$. Prove, using Turán's theorem, that there are at most $\lfloor \frac{n^2}{4} \rfloor$ pairs $\{i, j\}$ satisfying $|a_i + a_j| < 1$.

**Problem 4:** Let $X$ be a set of $n$ points in the plane with no two points of distance greater than 1. Show that there are at most $\frac{n^2}{3}$ pairs of points in $X$ that have distance greater than $\frac{1}{\sqrt{2}}$.

[Hint: .seerged 90 tsael ta elgna na htiw elgnairt a si ereht stniop ruof gnoma taht wohS]

**Problem 5:** Let $G$ be a triangle-free graph with $n$ vertices and minimum degree (strictly) larger than $\frac{2n}{5}$. Show that $G$ is bipartite.

[Hint: .hparg eht fo tser eht dna C neewteb segde tnuoc dna C elcyc ddo tsetrohs a redisnoC]

**Problem 6:** Prove, using the Kővári–Sós–Turán theorem, that between any collection of $n$ distinct points on the plane, there are at most $cn^{3/2}$ pairs that are of distance 1, where $c$ is some constant.