Graph Theory

Solutions 14

Problem 1: Let H be an arbitrary fixed graph and prove that the sequence $\frac{\mathrm{ex}(n,H)}{\binom{n}{2}}$ is (not necessarily strictly) monotone decreasing in n.

Solution: We will use an averaging argument to solve the problem. Let G be an extremal H-free graph on $[n]$ (i.e. G has $\mathrm{ex}(n,H)$ edges). Define $G_i = G - i$ for any vertex $i \in [n]$ and m_i to be the number of edges in G_i.

Of course, G_i is H-free, so $m_i \leq \mathrm{ex}(n-1,H)$. On the other hand, any edge of G is in exactly $n-2$ of the G_i (all but the two obtained by deleting an endpoint of the edge). This means that

$$(n-2)\cdot \mathrm{ex}(n,H) = \sum_{i\in[n]} m_i \leq n \cdot \mathrm{ex}(n-1,H).$$

Dividing both sides by $n(n-1)(n-2)/2$, we obtain $\frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \leq \frac{\mathrm{ex}(n-1,H)}{\binom{n-1}{2}}$, which is what we wanted to show.

Problem 2: Imitate the proof of Turán's theorem to show that among all the n-vertex K_{r+1}-free graphs, the Turán graph $T_{n,r}$ contains the maximum number of triangles (for any $r, n \geq 1$).

Solution: We go along the lines of the proof of Turán's theorem. By induction on r, we show that for every n, any K_{r+1}-free graph on n vertices has at most as many triangles as the Turán graph $T_{n,r}$. Clearly, for $r = 1, 2$ the statement is trivial. Now consider larger r. Let G be any K_{r+1}-free graph on n vertices. For a vertex v, let $m_G(v)$ denote the number of triangles containing v. Choose a vertex w where this number is maximal in the graph. Let $S = N_G(w)$ and $T = V(G) \setminus S$. Now, define a new graph H which consists of a Turán graph $T_{|S|,r-1}$ on S and a complete bipartite graph between S and T (but T is independent).

We claim that H contains at least as many triangles as G. We distinguish two types of triangles. Let $t_1(G), t_1(H)$ denote the number of triangles intersecting T, and let $t_2(G), t_2(H)$ denote the number of triangles inside S. We will show that $t_1(G) \leq t_1(H)$ and $t_2(G) \leq t_2(H)$, which implies the claim. First, since S is the neighbourhood of w in G and G is K_{r+1}-free, we know that $G[S]$ is K_r-free. Hence, by induction, $t_2(G) \leq t_2(H)$ since $t_2(G)$ is the number of triangles in $G[S]$ and $t_2(H)$ is the number of triangles of the Turán graph $T_{|S|,r-1}$. Now, consider the triangles intersecting T. The crucial observation is that the number of triangles containing vertex v is equal to the number of edges induced by the neighbourhood of v. In H, every vertex in T has neighbourhood S. Moreover, by Turán's theorem, we know that $e_H(S) \geq e_G(S)$ since

$G[S]$ is K_r-free and $H[S] = T_{|S|,r-1}$. Finally, $e_G(S) = m_G(w)$. We conclude that, for every $v \in T$, we have $m_H(v) = e_H(S) \geq e_G(S) = m_G(w) \geq m_G(v)$, where the last inequality holds by the choice of w. Finally, we conclude that

$$t_1(H) = \sum_{v \in T} m_H(v) \geq \sum_{v \in T} m_G(v) \geq t_1(G).$$

Note that the last inequality is not necessarily an equality as triangles inside T are counted more than once. However, since T is independent in H, the first equality holds as every triangle in H intersecting T is only counted once. This completes the proof of the claim.

Note that H is a complete r-partite graph $K_{n_1,\ldots,n_r}$. Observe that the number of triangles in $K_{n_1,\ldots,n_r}$ is $\sum_{1 \leq i<j<k \leq r} n_i n_j n_k$. We claim that for $n = \sum_i n_i$ being fixed, this number is maximized when $|n_i - n_j| \leq 1$ for all i, j. Suppose for a contradiction that this is not the case. Wlog, assume that $n_1 - n_2 \geq 2$. Define $n_1' = n_1 - 1$ and $n_2' = n_2 + 1$ and $n_i' = n_i$ for all $i \geq 3$. Then

$$\begin{aligned}
&\sum_{1 \leq i<j<k \leq r} n_i' n_j' n_k' - n_i n_j n_k \\
&= (n_1' - n_1) \sum_{3 \leq j<k \leq r} n_j n_k + (n_2' - n_2) \sum_{3 \leq j<k \leq r} n_j n_k + (n_1' n_2' - n_1 n_2) \sum_{3 \leq k \leq r} n_k \\
&= (n_1' n_2' - n_1 n_2) \sum_{3 \leq k \leq r} n_k > 0,
\end{aligned}$$

where the crucial inequality is that $n_1' n_2' - n_1 n_2 = n_1 - n_2 - 1 > 0$ by assumption. This completes the proof.

Problem 3: Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be vectors such that $|a_i| \geq 1$ for each $i \in [n]$. Prove, using Turán's theorem, that there are at most $\lfloor \frac{n^2}{4} \rfloor$ pairs $\{i, j\}$ satisfying $|a_i + a_j| < 1$.

Solution: Define the graph G on $[n]$ where $i \sim j$ iff $|a_i + a_j| < 1$. It is enough to show that G is triangle-free. But this is indeed the case, since for any $i, j, k \in [n]$,

$$|a_i + a_j|^2 + |a_j + a_k|^2 + |a_k + a_i|^2 = |a_i + a_j + a_k|^2 + |a_i|^2 + |a_j|^2 + |a_k|^2 \geq 3,$$

so at least one of $|a_i + a_j|^2, |a_j + a_k|^2, |a_k + a_i|^2$ is at least 1.

Problem 4: Let X be a set of n points in the plane with no two points of distance greater than 1. Show that there are at most $\frac{n^2}{3}$ pairs of points in X that have distance greater than $\frac{1}{\sqrt{2}}$.

Solution: Observe, first, that among any 4 points in the plane some three form a non-acute triangle (with an angle of at least 90 degrees). Indeed, if the points are in convex position, then one of the four angles of the quadrilateral is at least 90 degrees (their sum is 360), whereas if their convex hull is a triangle, then one (even two, in fact) of the three angles at the point in the middle is at least 90 degrees (again, their sum is 360).

Now look at the set X, and draw an edge between two points if their distance is greater than $1/\sqrt{2}$. We claim that this graph contains no K_4. Suppose it did, and look at the four points inducing a K_4. By the above claim, three of them form a non-acute triangle. Let us denote the sides by a, b, c, where the angle between a and b is at least 90 degrees. Both a and b have length greater than $1/\sqrt{2}$, so c must have length greater than 1, contradiction.

So this graph we defined is K_4-free, hence, by Turán's theorem, it contains at most $n^2/3$ edges, what we wanted to show.

Problem 5: Let G be a triangle-free graph with n vertices and minimum degree (strictly) larger than $\frac{2n}{5}$. Show that G is bipartite.

Solution: Suppose by contradiction that G is not bipartite, and let $C = (v_1, \ldots, v_k)$ be a shortest odd cycle in G. Then C has no chords (because else we can get a shorter odd cycle). We have $k \geq 5$ because G is triangle-free. We have $\sum_{i=1}^{k} d(v_i) \geq k \cdot \delta(G) > 5 \cdot \frac{2n}{5} = 2n$. The sum $\sum_{i=1}^{k} d(v_i)$ counts the edges inside $V(C)$ twice, and the edges between $V(C)$ and $V(G) \setminus V(C)$ once. The only edges inside $V(C)$ are the edges of C (because C has no chords), and there are k of them. So $e(V(C), V(G) \setminus V(C)) > 2n - 2k = 2(n-k)$. This means that one of the vertices in $V(G) \setminus V(C)$ must have more than 2 (so at least 3) edges to $V(C)$. Let $x \in V(G) \setminus V(C)$ be such a vertex.

The idea now is that if we have an odd cycle and a vertex outside the cycle which has 3 neighbours on the cycle, then we can find a shorter odd cycle. This will contradict the minimality of C. An easy case is when C is a cycle of length 5; it is easy to see that if a vertex has 3 neighbours on C then there is a triangle.

Suppose x is connected to v_i, v_j, and let P, Q be the two v_i, v_j-paths along C. One of P, Q has even length and the other has odd length (because C is odd). Suppose P is even and Q is odd. If $|P| > 2$, then the path v_i, Q, v_j, x, v_i (obtained by adding to Q the edges $v_i x, v_j x$) is an odd cycle shorter than C.

So if there is no shorter odd cycle, then $|P| = 2$, meaning that v_i, v_j are at distance 2 along C. Now suppose that x has three neighbours v_i, v_j, v_k on C. Then any two of v_i, v_j, v_k are at distance 2 on C. This means that C has exactly 6 vertices (i.e. v_i, v_j, v_k, an one vertex between each pair from v_i, v_j, v_k). But C was an odd cycle, contradiction.

Problem 6: Prove, using the Kővári–Sós–Turán theorem, that between any collection of n distinct points on the plane, there are at most $cn^{3/2}$ pairs that are of distance 1, where c is some constant.

Solution: Define a graph G on the point set, where two points x and y are connected by an edge if their distance is 1. It is easy to see that G does not contain $K_{2,3}$ as a subgraph. Indeed, the neighbours of each point lie on a circle of radius 1, and since two circles have at most two points in common, two of our vertices can have at most two common neighbours. Now an application of the Kővári-Sós-Turán theorem on the graph G gives the desired bound on the number of edges.